University of Southampton
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Making Progress

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Preface

Making Progress is the second of three sections which comprise an introductory-level university course module, *Routes to Success*.

Making Progress has been presented through a series of five or six weekly one-hour lectures. Students are expected to carry out a number of tasks, working together in small groups where appropriate, and to submit their work in portfolio form. The material in this document is not intended to be circulated to students as it stands: at the University of Southampton it has been presented in stages through a virtual learning environment.

Later editions of *Making Progress* are likely to appear in the future. Users are therefore advised to acquire the most recent edition available at any time.

1. Making Progress

This section of the module is about the progress you should be making as you work through the Foundation Year. Problem-solving is the main focus in this section, because an ability to solve problems will be of central importance to you if you are to succeed in the Foundation Year and in the subsequent years of your degree programme.

But there are other aspects of "making progress" besides problem-solving, and we will mention a few of them here:

- Academic courses in science and technology are generally progressive, with each part dependent on your understanding of the ideas presented in earlier parts. It may seem obvious, but to make good progress you should always make sure (by repeated revision, if necessary, even if no examination is imminent) that you understand what has already been covered, as well as what is being covered now.
- Know what's coming. It can make a big difference if you peek ahead through the lecture notes. It isn't necessary to learn things in advance - just having a feeling for what is coming will help. And look at forthcoming problem sheets, or at past examination papers, and think about what you would need to know to answer the questions there.
- Find answers to the questions that crop up in your own mind. If there is anything you don't understand, don't just shrug it off. Get to the bottom of it. The questions nagging at the back of your own mind are more important than the ones given to you on the problem sheets!

This module is assessed through a portfolio of work which you must submit in three stages. This section of the module (*Making Progress*) is the second stage. There are eight different items that you must submit, and details of them are given elsewhere here together with indications of what percentage of your final mark for this section depends on each piece of work. Also, you are asked to reflect on what you have done and write comments on the portfolio form itself; and 15% of the marks (for *Making Progress*) will depend on that.

2. Progress through Problem-Solving

There are various ways of learning things, but problem-solving is an especially important form of learning in engineering, science and mathematics. And not only is it a way of learning about a specific subject, it is also a valuable general skill - a "transferable" skill - because once you have become good at solving problems in astronomy, say, the same mental skills will help you to solve problems in other fields such as accountancy or marketing even though the subjects may be superficially very different.

Furthermore, learning is often ***assessed*** through problem-solving. A conventional examination paper, after all, consists of a series of problems which the examinee must solve. Performing well at examinations, therefore, requires ability and confidence in problem-solving.

In what follows, we will focus on various aspects of problem-solving including the use of numbers, words and images.

Just for fun, here is a little problem to start with:

A flesker has a mass of 125 tonnes and a density similar to that of water.

Estimate its surface area.

3. Numbers

3.1 Using Numbers

Although the basic rules of arithmetic are well known to anybody who enters a university, many students lack practice at handling numbers, especially to quantify unfamiliar concepts in contexts beyond everyday life. We all know the height of a ceiling and the mass of piece of cheese, but we don't know the diameter of the Sun or the charge of an electron. And units can be confusing: we know about seconds and kilograms, but have less of a feeling for light-years and newtons. (We know the mass of a piece of cheese, but what is its weight?) All these quantities obey the simple rules of arithmetic, but for some reason it seems hard to work things out and get the right answers.

Hand-held electronic calculators are part of the problem. Most of the calculations you will have to do during your university career can be done without a calculator. Most professional scientists and engineers do not use hand-held calculators (although there was a time when they did, about 35 years ago, before the advent of desktop computers). When students use calculators, it is noticeable that they often use them for working through several steps in succession, without writing down the intermediate results; and the calculator is often relied upon to keep track of powers of ten when very large or very small numbers are involved. Unless you are very familiar with your particular calculator, and you have taught yourself to use it with great skill, it is likely that using a calculator will cause you to make mistakes. And even if you can use a calculator accurately and reliably, there is a danger that, by never doing calculations in your head, or with pen and paper, you will forget how to do so - or, at any rate, you will fail to gain adequate practice. It is better not to use a calculator at all unless it is unavoidable.

You should not use a calculator in any of the exercises and activities following.

Here is a little calculation that you should be able to do in your head:

What is the cube root of the number you get by reversing the digits of the sum of the cubes of 2 and 4?

3.2 Units

Two things are needed to specify any quantity: the **number** and the **units**. For example, a distance might be specified as 15 (the number) kilometres (the units). To give the number by itself is meaningless: to say that the distance between the Earth and the Sun is 556 tells us nothing.

So, it is important to get into the habit of always linking a number to the units to which it refers. Some units are simple, but some are compound: an example of a compound unit is a "metre per second per second", the unit of acceleration. Getting the units right is important not only in science, but in all parts of life - if you are offered an apartment at a rent of 600, you should ask what the units are: they might be "dollars per calendar month" (another example of a compound unit) or "pounds per week" or something else, and you need to know.

It is a good habit never to write down a number without also writing down the units to which it refers.

(But is it really true that every number refers to particular units? Are there contexts in which we use numbers without units?)

3.3 Numbers

Mathematically there are many different kinds of numbers, but for our purposes we need only think of two, namely "integers" (whole numbers) and "real" numbers. Real numbers can have fractional parts to arbitrary precision (e.g. 58.5871302387045) and in many cases a real number may be impossible to represent accurately without an infinite string of decimal digits.

Do you know what an "irrational" number is? Why are they so called?

We can prove that the square root of 2 is irrational: here is the proof:

Irrational Numbers: the case of $\sqrt{2}$

(1) Suppose that $\sqrt{2}$ is rational. That means that $\sqrt{2} = p/q$ where p and q are integers.

(2) We may say that p/q is a fraction "in its lowest terms", i.e. that all common factors shared between p and q have been cancelled out. So, p and q have no common factors.

(3) Squaring the equation in (1) above, $2 = p^2/q^2$ therefore $2q^2 = p^2$ and therefore p^2 is an even number.

(4) p^2 has all the same factors as p, but with each factor occurring twice as many times. So, every factor of p^2 is a factor of it at least twice over. So, if p^2 is divisible by 2, it is divisible by 4, and therefore p is divisible by 2.

(5) Since, then, p is an even number, let $p = 2r$

(6) We may therefore replace $2q^2 = p^2$ by $2q^2 = 4r^2$ and therefore $q^2 = 2r^2$ and therefore q^2 is even, and therefore q is even.

(7) We have proved that both p and q are even. But our original hypothesis was that p and q had no common factors. So, we have a contradiction.

(8) It follows that we must have been wrong to suppose that $\sqrt{2}$ could be expressed as a ratio of two integers. Instead, $\sqrt{2}$ must be "irrational".

This is important not only because it shows how the existence of a new kind of number - not representable as a ratio of two integers - was first demonstrated: it is important also because it illustrates a particular technique for proving something. The technique is to start by assuming the **opposite** of what you are trying to prove, and then showing that the assumption leads to a contradiction.

Notice that a quantity which has particular units will normally be measured by a real number, not an integer. This does not mean that we cannot talk about, say, "a mass of 7 kilograms", but a mass may naturally take on any real-number value. We should understand "7 kilograms" to be a kind of approximation and we might talk about "7 kilograms" when the mass is actually closer to 6.95 kg or 7.3 kg.

For the sake of clarity, when writing down a real number, it is a good idea to give as many decimal places as correspond to the precision with which the number is actually known. For example, "7.30 kg" does not mean that the mass is *exactly* 7.30 kg. It just means that the mass is closer to 7.30 kg than it is to either 7.29 kg or to 7.31 kg.

This could be written "7.300 ± 0.005 kg". This notation, using the plus-or-minus sign to indicate the precision with which a number is known, is conventional whenever the reader is likely to need to know exactly how accurate the "7.30 kilograms" is. It is particularly important when the number is an experimental measurement, or is a result derived from experimental measurements: in that case the precision is often called the "error" in the number. The word "error" can be confusing: if we say that an object has a mass of 7.30 kg with and error of 0.005 kg, it does not mean that somebody has made a mistake! It just means that there is a likely margin of uncertainty of 0.005 kg above or below the value of 7.30 kg.

Sometimes it is useful to think in terms of "fractional" or "percentage" precision. For example, in the example we have been using, we might say that the 7.30 kg is known to a precision of 0.1%. (The precision itself does not have to be expressed to high precision! The "0.1%" is expressed to one significant figure.)

However, if you are asked to solve a problem with a numerical answer, you are not normally expected to give the error or uncertainty as well as the number itself. You are expected simply to give the answer with an appropriate number of decimal places. If the numbers which you are given at the start of the problem are all given to two decimal places, then you can assume that the final result will have a similar precision and can also be given with two decimal places. For example, if a problem involves taking the acceleration due to gravity to be 9.8 m s^{-2}, then it would be foolish to give the answer to six places of decimals.

3.4 Estimating

A few examples:

(1) If a number is given as 9.5 ± 0.5, what percentage precision is represented by that?

The answer is 5%. To be pedantic about it, we could say 5.3%, but 5% is good enough and there is not much point in going to another place of decimals when the "0.5" in the question is only given to one significant figure.

(2) There are 41 people in the room, and about 30% of them are women. How many women are in the room, and with what precision do we know that number?

We should not declare that there are 12.3 women in the room! Like most questions to do with precision, this really comes down to common sense. The phrase "about 30%" could mean anywhere between, say, 27% and 33% (because otherwise it might have been said that there were "about 25%" or "about 35%" instead of 30%). On that basis, there are probably from 11 to 14 women in the room, with the most likely whole number being 12.

It would therefore be sensible to say that there are 12 ± 2 women in the room.

By the way, an expression like "12 ± 2" does NOT mean that there cannot be more fewer than 10 or more than 14. It just means that the number PROBABLY lies in the range 10-14. There is still a possibility that the actual number might be outside that range: 6, say, or 19. To understand this fully, you would need to learn more statistics (Poisson distributions) but that is not necessary at this stage.

(3) Write to a precision of 0.1%:

8.297419
1.091825
732.81
911,964

Answers:

8.30
1.092
733
912,000

What we do is "round" the numbers to the number of significant figures which gives us a 0.1% (one-in-a-thousand) precision.

(4) Estimate the values of:

342 x 820
103/17
the square root of 58

Answers:

280,000
5.9
7.5

There are often several alternative ways of tackling questions like these, and the answers given are not the only "correct" ones. For 342 x 820, you can just raise 342 slightly and at the same lower 820 by a similar proportion, giving, say, 350 x 800, which a mental calculation gives as 280,000.

For 103/17, you can raise 17 slightly and also raise 103 slightly, in the same proportions. Raising 17 by 3 makes 20, and is roughly a 15% increase. Raising 103 by 15% makes about 103 + 15 = 118. So we just divide 118 by 20, which can be done mentally, giving 5.9.

To estimate the square root of 58, just notice that it is intermediate between 49 (of which the square root is 7) and 64 (of which the square root is 8). So, the answer is probably not far from 7.5.

(4) Estimate the value of the square root of π.

To start with, remember that π is equal to 22/7 to a fair precision.

So, the problem is to estimate the square root of 22/7. A good way of proceeding would be to multiply the top and the bottom of this fraction by 7 so that the denominator is a perfect square, and we have

(22 x 7)/7 x 7)

which is equal to 154/(7 x 7). To estimate the square root of 154, notice that it lies between 144 (the square of 12) and 169 (the square of 13). So, roughly, the square root of 154 will be about 12½. Our estimate for the square root of π is therefore 12½/7, which is close to 1.8.

3.5 Exercise: how far away is the Moon?

Outline:

This exercise does not involve the use of any special equipment. The idea is for you to estimate first the size of the Moon and then its distance from the Earth. In doing this, it is important that you should appreciate the uncertainties in what you are doing, and estimate the likely errors in the results that you obtain.

Procedure:

(a) You may need to start by discussing what happens when a lunar eclipse takes place. Although it is possible for this to be done by one person working alone, it is better for a small group to work together to thrash out what measurements and calculations are required.

(b) You are provided with three images of lunar eclipses (below). You will need to make measurements from these images, using a ruler on your computer screen or on printouts of the images. (What exactly should you measure? Think hard about what we see when we look at an eclipse of the Moon.)

(c) The kilometre was originally defined as 1/10000 of the distance from the North Pole to the Equator through Paris.

(d) From (b) and (c) above, you should be able to estimate the radius of the Moon. Actually, since you have been given three different images of lunar eclipses, you should start by making three independent estimates of the radius of the Moon. Then, you should combine them to arrive at a single best estimate, quoting it with an appropriate level of precision and stating the likely magnitude of the error in it.

(e) When there is a total eclipse of the Moon, the period of totality (during which the Moon is entirely covered by the Earth's shadow) can last for up to a maximum of 1 hour and 47 minutes. The Moon circles around the Earth roughly once every lunar month. Using these two facts, and using your values for the radii of the Moon and of the Earth, you should be able to work out the distance of the Moon. (Hint: you must draw a diagram, and you might find it useful to think about the <u>angular</u> diameters of the Moon and of the Earth's shadow as seen from the Earth.) This method was used by the ancient Greek astronomer Aristarchus. Again, quote your result with appropriate precision and with a statement of the likely error in it.

And finally:

Your report on this exercise need not be lengthy but you should describe exactly what you did, record the measurements and calculations you made, and draw diagrams where appropriate to make it all clear to the reader. In your final summary, gather together the two results you should have obtained (the radius and distance of the Moon) stating them with reasonable precision and correct units with their likely errors.

Draw any other conclusions that you think fit, and comment on any particular difficulties that you encountered with this activity.

The images of lunar eclipses which you will need to use for this exercise can be found here:

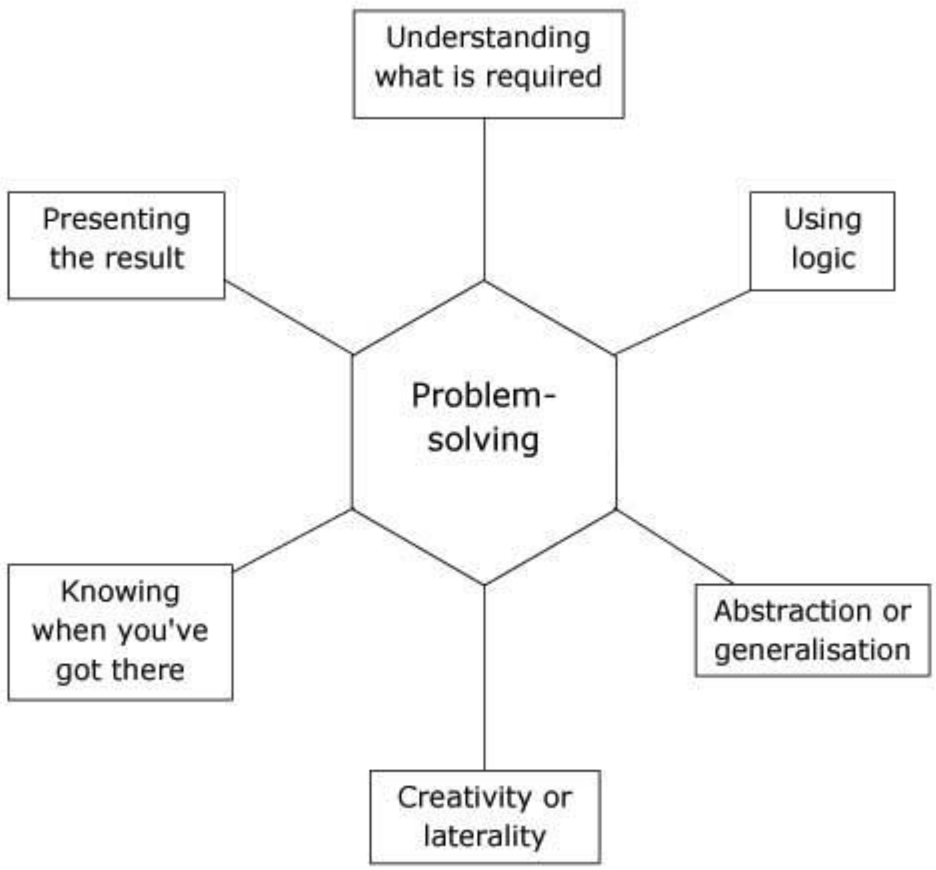
Image 1

Image 2

Image 3

3.6 Aspects of Problem-Solving

A diagram to reflect on:



4. Words

4.1 Understanding Words

Although a flair for mathematics is important for any scientist or engineer, it is also vital to be able to understand **words** properly. Many of the difficulties which people have with science are really problems with the use of words. No less so than a lawyer or historian, a scientist needs to be able to analyse the meaning of a written text and to be able to write with precision and conciseness.

Very often, a student who cannot solve a mathematical problem turns out not to have understood what was asked. The first step in solving any problem is to understand exactly what the problem means, and what situation or scenario the question-writer has in mind. Perhaps surprisingly, this can be a difficulty for those whose mother language is English just as much as for those brought up to speak other languages. In fact, curiously enough, native English-speakers may be at a disadvantage: they will often jump to a hasty conclusion about the meaning of a sentence, while a non-native-speaker is likely to analyse the sentence more systematically.

It is surprisingly easy to misinterpret a question. For somebody writing a question, it is easy to be unintentionally ambiguous. (For example, "Who was elected President of the United States in 2000?")

4.2 Who is the English Prime Minister?

The following was written while Gordon Brown and the others mentioned were in power, and should be modified when they have left office!

When you see a question, read it carefully and work out exactly what is required. Very often, you will find that a question is ambiguous or misleading. For example, the question "Who is the English Prime Minister?" could be responded to in any of the following ways:

- Gordon Brown
- There isn't an English Prime Minister, because Gordon Brown is not English. (He is a Scot.) The last English Prime Minister was John Major.
- The question is wrongly phrased, because there is no Prime Minister of England. It should read "Who is the British Prime Minister"?

Notice that the second and third responses above are based on different interpretations of what is meant by the phrase "English Prime Minister". The word "English" could be an adjective of nationality describing a person who is a "Prime Minister"; or it could be an abbreviation of the phrase "Prime Minister of England", in which case it might be applicable to a Welshman. It is conceivable that the only "English Prime Minister" at the present time might be the Prime Minister of, say, Australia (if Kevin Rudd happened to have been born in England, which he was not!)

So, let's rephrase the question:

4.3 Who is the British Prime Minister?

This version of the question could attract either of the two following responses:

- Gordon Brown, because he is British. But some people might argue that Brian Cowen is also "British" because, technically, Ireland is part of the "British Isles": from that point of view there are two British Prime Ministers and the question is ambiguous.

- The question is wrongly phrased, because "Britain" is not the title of a nation with a Prime Minister. It should read "Who is the Prime Minister of the United Kingdom?"

So let's try that....

4.4 Who is the Prime Minister of the United Kingdom?

Well, actually, it is still not entirely clear what sort of answer is expected. Here are three possible answers:

- The Prime Minister of the United Kingdom is the leader of the party, or coalition of parties, which commands a majority in the House of Commons.

- In the United Kingdom, the Prime Minister is the executive head of the government, answerable to the monarch, and is the Kingdom's principal representative in its dealings with other countries.

- The current UK Prime Minister is Gordon Brown.

All three of these statements are, more or less, correct. But they represent different interpretations of what the questioner wanted to know. The first reply is rather like a "person specification", saying what qualifications are needed for the job of Prime Minister. The second reply is like a "job specification", saying what a Prime Minister has to do. The third reply tells us who the current PM is, but that might not have been what the questioner wanted to know. We see, therefore, that it is possible to misunderstand the meaning of as simple a word as "who"!

4.5 Translating Words into Mathematics

We often need to be able to turn a statement given in words into an equivalent statement in mathematical form. For example, you can try to express the following in mathematical form:

- My house is more than twice as wide as it is deep.
- The area of a rectangle is intermediate between the areas of the squares that can be drawn on its sides.
- The area of a rectangle is the average of the areas of the squares that can be drawn on its sides.

These might seem very simple exercises, but they are often found to be surprisingly difficult by students who have not tried this sort of exercise before!

Notice, by the way, that you can try to express a statement in mathematical form whether or not the statement is true. The first statement above may or may not be true. The second statement is always true, and expressing it in mathematical form might be the first step you would take towards proving it. The third statement is not generally true, and expressing it mathematically might be the first step in proving it to be generally false.

4.6 Understanding Definitions

Definitions are extremely important in science and technology. Any experiment, debate, proof, or design must start with clear agreed definitions of the terms used. Science can be regarded as a logical structure whose foundations are the definitions of the most fundamental scientific concepts.

What is this a definition of?

XXX is a function of location in space and time, such that the difference between the XXX's at two points is equal to the electrostatic energy difference between a coulomb of electricity at one of the points and a coulomb of electricity at the other.

And this?

YYY is a vector function of two locations in space, and its components are the differences between the spacial coordinates of the two locations.

It is sometimes difficult to understand the definition of something, but it is generally even more difficult to compose a definition. When you write down a definition of something, it should...

- make grammatical sense.
- be concise.
- **not** be a list of examples of the thing defined.
- be unambiguous.
- be based on more (not less!) fundamental concepts.
- not lead to circularity when read alongside other definitions (e.g. defining voltage in terms of current and resistance, defining current in terms of resistance and voltage, and defining resistance in terms of voltage and current, which leaves nobody any the wiser!)
- be comprehensive, covering all instances of the thing being defined.
- be true!

If you try hard to define **electrical resistance** or **coefficient of static friction**, you will see how difficult it can be to arrive at a truly satisfactory definition. But the effort to do so pays off, because to carry out the task of writing a definition you have to get the concept completely straight in your own mind!

4.7 Definitions of electrical resistance and static friction

The electrical resistance between two points is the voltage difference which would need to be sustained between the two points in order to cause a direct electric current of one ampere to flow between them.

(Notice that this is only one of various alternative definitions that could be given. It is a quantitative definition, encapsulating Ohm's Law. But, for the sake of conciseness, this definition does not extend itself to include an definition of the "ohm".)

The coefficient of static friction between two materials is the maximum reactive shear force between adjacent parallel surfaces of the two materials, expressed as a proportion of the compressive force between them, in circumstances where the reactive shear force counteracts an externally-imposed shear force and therefore prevents lateral movement between the surfaces.

This definition is not perfect, and it depends on some other more basic concepts (such as a "reactive" force) but it is not far from the mark.

4.8 Words: a definition exercise

Please produce a definition of one of the following, according to the letter code that you are given:

A. Alternating Current
B. Temperature
C. Kinetic Friction
D. A Cycloid
E. Frequency
F. Tensile Stress
G. Pressure
H. An angle
J. Gas

Your definition should follow the guidelines listed above under "Understanding Definitions". Moreover, your definition should be verbal (i.e. without mathematical formulae) and should be printed, not handwritten. It is to be included as part of your portfolio submission for this part of the module.

5. Problems

5.1 Analysing Problems

There is not just one way to solve a problem. There are various different approaches, appropriate to different circumstances. The tools that we use to solve problems include

- Basic scientific principles expressed as standard formulae. ("What's the formula for it?")

There are also standard mathematical processes such as

- Simplifying expressions
- Eliminating uninteresting variables
- Inserting numbers into algebraic expressions

including processes which will probably not be so familiar to you at this stage:

- Differentiating something
- Integrating something

and deciding on the "something" can be the hardest bit of the problem.

Logic, naturally, is a vital problem-solving tool. Examples are

- Induction
- Deduction
- *Reductio ad absurdum*
- Arguments from symmetry

and examples of some of these are mentioned below.

5.2 The Solution as a Work of Art

Solving a problem is not just a matter of finding "the answer". It is true that the final numerical result is often the most important output from a problem-solving exercise, and if you can arrive at that final result then it reassures the teacher that you know what you are doing. In professional life, however, it is usually expected that the "solution" to a problem will be a presentation of the whole chain of calculation that led to the final result.

Presenting a solution to a problem is a very important form of communication, and it should enable the reader to share in your reasoning and to see exactly how the initial data are used to obtain the eventual final answer. The skill of presenting a solution is a form of artistry which goes well beyond merely recording your own mathematical working.

As an individual exercise, you are asked to produce a solution to a selected problem. What you must actually submit is a printed copy of a Word document. The equations must be formed using the Word equation editor, and drawings must also be done using Word - not by hand.

The point is not simply to show that you can find a solution to the problem. The point is to demonstrate that you can set out a calculation clearly and logically, without using a calculator, with appropriate words and diagrams to explain what you are doing, dealing properly with the units and giving results with the appropriate precision.

You may wonder why you must present it in this form. In the past, software for equation-writing has been generally rather clumsy and tricky to use, and universities have almost invariably expected students to submit mathematical work in handwritten form. But things are changing, and you should find that a modern version of Word, when you get used to it, is quite quick and effective for displaying mathematics. You might like to know that the Open University has recently been experimenting with the online submission of mathematical coursework in electronic form, with very positive results. Moreover, by using Word you will be able to produce an attractive, professional-looking result, easy for somebody else to understand, in which you will be able to take pride regardless of the quality of your handwriting.

Your solution should include details of how you checked your answer to make sure it was correct. This "Solution as a Work of Art" will account for 20% of the marks for *Making Progress*.

Choose the problem from the following two options:

(1) Calculate the diameter and circumference of a circle if the area of a sector which subtends an angle of $1{\cdot}45$ rad is 620 mm^2.

(2) A train accelerates uniformly from rest to reach 60 kilometres per hour in 6 minutes, after which the speed is kept constant. Find the total time taken to travel a total distance of 6 kilometres.

Remember, a good diagram (of the circle or of the velocity-time diagram) will be expected in either case.

5.3 Logic in Problem-Solving: 1

Reductio ad absurdum:

One way of proving that a statement is false is by

1. Assuming that it is true,
2. Making deductions based on that assumption, until you
3. find that you can deduce that the original statement is false.
4. Since a statement cannot be both true and false, it follows that the assumption you started from must have been incorrect, i.e. the original statement was false.

We have already seen (in the "Numbers" section) how to use this technique to prove that the square root of 2 is irrational.

Now, can you use the same technique to prove that the cube root of 3 is an irrational number?

5.4 Logic in Problem-Solving: 2

Induction:

Suppose a statement can be made with reference to any specific integer number n. (For example, the statement could be that "$n^3 - n$ is divisible by 3 when n = 7".)

Then imagine that you are asked to prove that the statement is true for ALL values of n.

The method of induction is a way of doing this.

Let us take a classic example: we will prove that the sum of the squares of the first n integers is equal to

$$n(n+1)(2n+1)/6$$

for all values of n.

We start by defining some terminology: let us call the sum of the squares of the first n integers S_n. Then,

$$S_{n+1} = S_n + (n+1)^2$$

This should be obvious! The next step is to **assume** that the formula we have to prove is true for the value n. We can therefore insert that formula into the above equation for S_{n+1}, giving:

$$S_{n+1} = n(n+1)(2n+1)/6 + (n+1)^2$$

This can be re-arranged with a little algebra, leading to:

$$S_{n+1} = (n+1)(n+2)(2n+3)/6$$

But this is just the same as the original formula for S_n, but with n replaced by (n+1).

So, we have shown that if the formula is true for an integer n, then it is also true for the next integer, n+1.

The final step is to show that the formula is true for the lowest possible value of n, i.e. n=1. This is very easy. Then, it follows that it must be true for ALL values of n.

Can you now prove that the sum of the first n integers (starting from 1) is always

$$n(n+1)/2?$$

And can you prove that $5^n - 1$ is divisible by 4 for all values of n?

5.5 A Tough One

Find a pair of distinct positive integers a and b such that

$$a^2 + b^2 - a.b$$

is a perfect square.

What sort of problem is this?

A very difficult approach is to try to analyse the problem in detail and find a general method of generating pairs of numbers with that property (e.g. by analogy with finding "Pythagorean" triangles)

A simpler approach is to devise a "search strategy" to look systematically for pairs (a, b) which might fit the bill. In fact, for anyone who has become adept with the Excel spreadsheet system, it would be fairly simple to set up a matrix of integers (a, b) and search for pairs for which the square root of the expression above is a whole number.

Sometimes a problem can be translated from one "language" into another. Compare the above expression to the Cosine Rule. Can you see that the question can be rephrased as "find an integer-sided triangle of which one angle is 60°"?

5.6 For your portfolio:

You should produce, in a properly-laid out form, your proof **either** that the cube root of 3 is irrational **or** that $5^n - 1$ is divisible by 4. You can choose which one. It should be presented as part of your portfolio for this part of *Routes to Success*.

5.7 The Matchstick Game

Two players, A and B, take turns in the following game. There is a pile of six matchsticks. At each turn, a player must take one or two sticks from the remaining pile. The player who takes the last stick wins.

If player A makes the first move and each player makes the best possible move, who wins?

The "problem" here is not just to say whether A wins or B wins. The problem is to understand the game fully, so that you can say what the general winning strategy is and who will win, however many matchsticks there are at the start. As a first step, it is a good idea to develop a way of displaying graphically the options open to each player at each turn. A "tree" diagram of some sort is appropriate. Once you have a good way of displaying on paper how the game can unfold, you are three-quarters of the way towards solving the problem.

6. Diagrams

6.1 Diagrams and Graphs

Drawing a diagram is a way of thinking!

Drawing an accurate diagram can help you to understand a problem much more deeply than if you just think about it verbally or use algebra alone to reach a solution.

In the classroom session on this topic we consider various types of diagrams that can be interesting to draw. It is surprising how difficult it can be to draw an apparently-simple diagram. An example is the "hodograph" of a point on the rim of a wheel of a moving bicycle, in other words the graph of the vertical component of its velocity (v_y) against the horizontal component of its velocity (v_x).

6.2 Exercise: triangle of forces

As an exercise to be included in your portfolio, please present a graphical solution to this question:

A roller of mass 70 kg is held in equilibrium on a smooth plane which is inclined at 30° to the horizontal. The roller is held by a cable. Find the force exerted by the cable, and the normal reaction of the plane on the roller, when the cable acts at an angle of 15° above the plane.

You should actually present two diagrams: (a) a clear diagram to illustrate the physical layout, and (b) a "triangle of forces" drawn to scale so that you can obtain the solution as accurately as possible by measurement from the diagram. You may need to use a ruler and a protractor. Having solved the problem graphically, you should NOT solve it by calculation.

6.3 Exercise: a hodograph

As an exercise to be included in your portfolio, draw the hodograph of the motion of the bob of a simple pendulum. Take it that the pendulum is swinging in the x-y plane, x being the horizontal and y the vertical direction. You have to plot v_y against v_x. Assume that the pendulum is swinging in a fairly wide arc (say, about 45° each way).

6.4 Exercise: graphs of functions

As an exercise to be completed for your portfolio, plot the graphs of any two of these relationships:

(1) $| x + y | = 1$

(2) $x^{100} + y^{100} = 1$

(3) $4y^2 = 4 - x^2$

(4) $| x | + | xy | + | y | = 1$

(5) $\sin x = \cos y$

(6) $x^2y^2 = 16$

In these equations, x and y are the usual Cartesian coordinates.

7. Is the Answer Right?

7.1 Checking the Answer

When a problem has a numerical answer, it is a good idea to somehow check it after having calculated it. Even when a problem has an algebraic answer, there are ways in which you can check on the plausibility of your result. Below, various ways of checking a result are mentioned.

Also, we look in detail at the method of "dimensional analysis".

7.2 "Casting out the Nines"

From any integer, it is possible to calculate a "check digit" or "digital root" which can be used to check for errors in calculations. All you have to do is add together all the digits of the integer, and then add them together again and again until the original integer has been reduced to a single digit.

For example, the number 5120 gives us the integer 8 (from 5+1+2+0), and the number 6739 gives us 7 (6+7+3+9 = 25, 2+5 = 7).

It is quite easy to calculate the digital root in your head because you can just work through the digits from left to right, always reducing their sum to a single digit. Taking the previous example again: 6+7 = 13; 1+3 = 4; 4+3 =7; 7+9 = 16; 1+6 = 7.

In fact, it is even easier than that, because whenever the digit 9 occurs (either in the original integer or when you add two digits together) it can simply be ignored. So for the integer 6739 we can ignore the final digit and the calculation is just 6+7 = 13; 1+3 = 4; 4+3 = 7.

Exactly why you can always ignore the 9's is easy to see if you try a few examples and think about it.

These digital roots (or "check digits") are useful because if we add any two (or more than two!) numbers together, then ***the digital root of their sum is the digital root of the sum of their digital roots.***

To take an example: suppose we have the addition

$$2790674 + 6128829 = 8919503$$

Then how can we check whether this is correct or not? We do it by forming the digital roots from 2790674 (which comes to 8), from 6128829 (0) and 8919503 (8). Then, because 8 + 0 = 8, the correctness of the sum is confirmed.

Of course, just because the digital roots check out correctly it does not mean that the calculation is necessarily correct: but if the digital roots do ***not*** tally then it tells you that a mistake was made.

This also works for multiplication (and for division and subtraction, too). Here is an example:

$$18977024 \times 37 = 702149888$$

Is this likely to be right? The digital root of 18977024 is 2, and that from 37 is 1. Multiplying them, we get 2 x 1 = 2. The number on the right, 702149888, gives us a digital root of 2, which matches. So, the calculation is not likely to have been wrong.

As an exercise, consider the calculation

$$816334947 \times 53 = ?$$

Imagine that you know that the answer is one of the following numbers:

44265752291
44235751291
43265752191

Which one is right?

7.3 Checking an Algebraic Expression

There are some effective techniques for checking the correctness of an algebraic expression. We can check the dimensions of the terms in the expression, we can check for symmetries, and we can check the values of the expression in simple special cases.

The Volume of a Frustum:

A square pyramidal frustum is a three-dimensional figure like a pyramid with its top chopped off. It has a square horizontal base and a square horizontal top, with a uniform height between them. (To see illustrations, put "frustum" into Google).

The ancient Egyptians were aware of a formula for the volume V of a square pyramidal frustum. V is a function of the lengths of the sides of the base (a), the length of the sides of the top surface (b) and the height of the frustum (h).

It would not be difficult for you to calculate the function V(a, b, h). But suppose you are presented with the following seven possible forms of V:

$$a(ha - ab + bh)/3$$

$$h^2(a^2 - ab + b^2)/3$$

$$h(a^2 + ab + b^2)/3$$

$$a(ha + ab + bh)/3$$

$$h(a^2 - ab + b^2)/3$$

$$h(a^2 - b^2)/3$$

$$h(a + ab + b)/3$$

Which of these could be correct? Without bothering to actually derive the function V(a, b, h) we can say:

- The second formula above has the dimensions of the fourth power of length. This must be wrong, because a volume is a third power of length (cubic metres, m^3, not m^4).

- The final formula in the list has, inside the brackets, lengths (a and b) and a product of lengths (ab) added together. This must be nonsense. You cannot add metres to square metres. You can only add together quantities with the same dimensions.

- The frustum is the volume between two squares separated by a height h. If you think about it, you can see that the volume is not changed if we turn the frustum upside down, replacing a by b and vice versa. So, the formula for V must be symmetrical under the interchange between a and b. However, the first, fourth and sixth formulae in the list above will become different if a and b are interchanged. (The sixth will be reversed in sign.) So, these formulae do not have the expected symmetry and must be wrong.

- That leaves us with the third and fifth formulae in the list. To check them, we can use the technique of considering what they will give us in a simple special case. A special case that springs to mind is when a=b=h, when the frustum is simply a cube. In that case, it is easy to see that the third formula has the value a^3, whereas the fifth formula has the value $a^3/3$. Since the volume of the cube is obviously a^3, we conclude that the fifth formula must be wrong.

- The arguments above leave us with only one plausible formula among the seven listed for the volume of a frustum, namely $V = h(a^2 + ab + b^2)/3$.

Of course, we have not actually proved that this ***is in fact*** the formula for the volume, but we have shown that none of the others can be right. For experienced mathematicians, scientists and engineers, checking expressions in this way becomes second nature.

7.4 Dimensional Analysis

In the section above, looking at formulas for the volume of a frustum, it was mentioned that you can check an algebraic expression for dimensional consistency. "Dimensional analysis" is a more systematic way of using dimensional consistency not only to check whether a formula might be right but, in some cases, to work out the formula in the first place.

Planets:

As a first example of dimensional analysis, consider the following question:

The rotation period (t) of a planet around the Sun depends on the planet's distance from the Sun (R), the mass of the Sun (m), and the gravitational constant ($G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$). Find the relationship between t and R. If Mars is 1.5 times as far from the Sun as the Earth is, how long is the Martian year?

The wonderful thing about dimensional analysis is that you can answer this question even if you have been taught nothing about gravity and have never heard of the gravitational constant!

We start by just saying that the time t must be proportional to some combination of powers of R, m and G multiplied together:

$$t \propto R^{\alpha}\, m^{\beta}\, G^{\gamma}$$

The symbol $\propto$ means "is proportional to". In this formula, the Greek letters α, β and γ represent unknown powers which could be positive or negative and could be whole numbers or fractions.

We now start out our dimensional analysis by writing the dimensions of each item in the formula. It is conventional to use a notation with M, L and T (or some combination of them) in square brackets to represent the fundamental dimensions of mass, length and time. So, written in that way,

the dimensions of t are [T]
the dimensions of R are [L]
the dimensions of m are [M]
the dimensions of G are $[L^3\ M^{-1}\ T^{-2}]$

and if the formula for t is to be dimensionally consistent, it follows that

$$[T] = [L]^{\alpha}\ [M]^{\beta}\ [L^3\ M^{-1}\ T^{-2}]^{\gamma}$$

We now rearrange the right hand side:

$$[T] = [L]^{\alpha+3\gamma} [M]^{\beta-\gamma} [T]^{-2\gamma}$$

Because the powers of each dimension must match, we conclude that

$$\alpha + 3\gamma = 0$$
$$\beta - \gamma = 0$$
$$-2\gamma = 1$$

which tells us that $\alpha = 3/2$, $\beta = -1/2$ and $\gamma = -1/2$. In other words,

$$t \propto R^{3/2}\, m^{-1/2}\, G^{-1/2}$$

or specifically (since we are not really interested in m or G)

$$t \propto R^{3/2}$$

This is the answer to the first part of the question - the relationship between t and R. Introducing a constant (q, say) this can be expressed as an equation

$$t = q \,.\, R^{3/2}$$

and, for the two planets Earth and Mars,

$$t_{EARTH} = q \,.\, R_{EARTH}^{3/2}$$

$$t_{MARS} = q \,.\, R_{MARS}^{3/2}$$

which means that

$$t_{MARS}/t_{EARTH} = (R_{MARS}/R_{EARTH})^{3/2}$$

We are told that $R_{MARS}/R_{EARTH} = 1.5$, and it is easy to calculate that $(1.5)^{3/2} = 1.84$. Since t_{EARTH} is, of course, one year, it follows that

$$t_{MARS} = 1.84 \text{ years}$$

This is the answer to the second part of the question. Notice that this has been worked out by a dimensional analysis which does not require any knowledge of the theory of gravitation. This ***has to be*** the answer simply because the quantities involved (t. R, m and G) cannot be related in any other dimensionally-consistent way.

The Simple Pendulum:

The most common textbook example of dimensional analysis is the period of oscillation of a simple pendulum. We start by noticing that the period (t) must depend on the length of the pendulum (X), the mass of the swinging bob (m) and the acceleration due to gravity ($g = 9.8\ m\ s^{-2}$).

So, with indices α, β and γ which will have to be calculated, we can write

$$t \propto X^{\alpha}\, m^{\beta}\, g^{\gamma}$$

giving the dimensional equation

$$[T] = [L]^{\alpha}\ [M]^{\beta}\ [L\ T^{-2}]^{\gamma}$$

which balances when $\alpha = 1/2$, $\beta=0$ and $\gamma = -1/2$. In other words,

$$t \propto (X/g)^{1/2}$$

As with all dimensional analyses of this sort, we just find a proportionality, not an equality. But it is enough to tell us that the period of oscillation of a simple pendulum is proportional to the square root of the length X of the string and that it does not depend at all on the mass of the bob.

7.5 Dimensional analysis: an exercise

A car of mass m is driven by a special kind of motor which always works at a constant power P. All the power goes towards increasing the speed of the car, with no loss of energy due to friction, air resistance or anything like that. The car starts from rest, and after a time t it has covered a distance x.

Use dimensional analysis to find how x depends on t, P and m.

If the car reaches a distance x = 100 metres after a time t = 10 seconds, how much further will it go in the next 10 seconds?

This calculation should be set out in detail and presented as part of your portfolio.

8. Portfolio Requirements

Here is a checklist of the portfolio requirements for the assessment of this part of the module:

- Numbers: the Distance of the Moon [20%]
- Words: a definition exercise [5%]
- Problems: an exercise in logic [10%]
- Problems: presenting the full solution to a problem [20%]
- Diagrams: example from a problem sheet [10%]
- Diagrams: the graphs of functions [5%]
- Diagrams: a hodograph [5%]
- Checking answers: dimensional analysis [10%]
- Comments on the portfolio form [15%]

It is not necessary to do these pieces of work in this order. In fact, it is expected that the two major exercises (presenting the full solution to a problem, and estimating the distance of the Moon) will be done over a period of time and will overlap some of the other tasks. However, the final task of completing the portfolio form should not be done until you have finished and collated all the other items.

9. More Problems for Fun

Many kinds of mathematical problems were reviewed during the preparation of this section of the course. Those below were not incorporated into the classes or assignments, and so you are not expected to solve them, but you might enjoy trying one or two of them anyway.

(1) A rectangle has a certain fixed perimeter. Show that its area is greatest if it is a square.

(2) Find two integers whose cubes add up to 250.

(3) Find two integers whose cubes add up to the square of 13.

(4) Prove that you cannot write the digits from 0 to 9, each once only, making an addition sum adding up to 100. (For example, 16 + 40 + 2 + 3 + 57 + 8 + 9 is not equal to 100.)

(5) If

$$f(x) = x^2 + 1$$

and

$$g(x) = x^2 - 1$$

solve the equation

$$f(g(x)) = g(f(x))$$

(6) Find a set of three different integers a, b and c such that

$$a^2 + b^2 = 2c^2$$

Solutions are provided at the end of the feedback section which follows.

10. Making Progress: Feedback

Below is an example of collective feedback that has been given to students for this section of *Routes to Success* (in addition to specific individual feedback).

General Feedback on Routes to Success (Making Progress):

The second part of the coursework portfolio for "Routes to Success" was generally done very well. There were many students who completed all the items required and gained marks in the 80s or 90s. However, there was a long tail of students whose marks were much lower: as a rule, low marks were obtained not because work was done badly but because some items of work were not done at all.

(a) General reflection: students were expected to reflect on the questions raised and to write clearly and perceptively. The majority of responded very well indeed to this.

(b) Reflection on solving the "Moon" problem: you should have shown that you understood the process that you had gone through. Most people did this very well.

(c) Finding the distance of the Moon: those who obtained the highest marks showed all their working, explained their methods clearly, drew good explanatory diagrams, and reached reasonable conclusions. Some people solved the problem successfully but presented neat but over-abbreviated reports. A few people solved the problem but presented their work so untidily that it was hard to tell that they had done so. Many people quoted their results with unrealistic precision. Almost everybody had a good stab at the problem.

(d) The definition exercise: this was generally well done, but sometimes definitions were insufficiently precise or were circular (for example, if you are defining "tensile stress" you not use the word "tensile" in the definition; and you shouldn't define "alternating current" as current which alternates).

(e) The exercise in logic: this was done very well by the great majority of students. Still, it was necessary to give explicitly every step in the argument, and one or two people lost marks by jumping steps.

(f) Presenting the solution to a problem: the idea was to display the solution to the problem to a high presentational standard, with the calculation being fully explained and illustrated electronically (i.e. using Word and its equation editor).

(g) Problem-solving with a diagram: here, the idea was to produce a good diagram of the physical configuration and then to solve the problem by drawing the triangle of forces and measuring the side representing the force to be determined. Few people did this completely satisfactorily. Some drew the triangle of forces but did not appear to obtain the result from it.

(h) Drawing the graphs of functions: a lot of people got full marks for this, but others lost a mark by showing only half of a graph (i.e. omitting the negative-y part). In some cases, completely wrong graphs were shown, either because of algebraic errors or because curve-drawing software was used incorrectly.

(i) The hodograph: well done by most people.

(j) Dimensional analysis: some people solved the thing almost completely but lost a mark by not stating the final answer of 182 m. Some people did the first part correctly (obtaining the formula) but then went wrong with the second part by using the standard kinematical formulae which are applicable only to constant-acceleration cases.

(k) Final reflection: usually very good.

(l) More Problems for Fun:

(1) Let the sides be a and b, the perimeter P and the area A:

$$P = 2a + 2b$$

$$A = ab$$

Now, let

$$a = P/4 + x$$

and

$$b = P/4 - x$$

which automatically satisfies $P = 2a + 2b$. Then,

$$A = ab = (P/4)^2 - x^2$$

which is obviously a maximum when $x = 0$, i.e. if $a = b$ and the rectangle is a square. (So you don't need calculus!)

(2) Read the question. It doesn't say that the two integers have to be different. The answer is 5 and 5.

(3) Read the question. It doesn't say that the two integers have to be positive. The answer is -7 and 8.

(4) Start by writing

$$S = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

and notice that $S = 45$. Now, any of these digits (n, say) may be shifted from being a unit to being a multiple of ten. For example, if $n=3$ we get

$$0 + 1 + 2 + 4 + 5 + 6 + 37 + 8 + 9$$

and as far as the sum is concerned it doesn't matter which other digit the n is tacked onto. The effect of shifting a digit like this is to reduce the sum by n (when n is removed as a unit) and then to increase it by $10n$ (when n is added to the tens). The net effect, therefore, is to increase the sum S by $9n$.

Since $9n$ is obviously a multiple of 9, shifting a digit from the units to the tens will therefore always increase the sum by a multiple of 9. However, the sum we started with, S, is itself a multiple of 9. So, any movements of digits from units to tens must always still leave us with a sum which is still a multiple of 9.

Since 100 is not a multiple of 9, the sum cannot be made equal to 100.

(5) $x = 1/\sqrt{2}$

(6) For example $a=1$, $b=7$, $c=5$. Rather than trying to find a mathematical procedure which will generate all possible solutions (which is possible but tough) you just need to follow a systematic search procedure or use Excel.

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Description	The second of two parts comprising “Routes to Success”, a module on study skills for an introductory university course oriented towards engineering, physics and geophysics
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Language	English
Filesize	Approx 400 KB (zipped file)
File format	Main file in PDF and docx formats plus one auxiliary file in docx format.