QM I: Examples: solutions

1. Use $\lambda = h/p = \frac{6.626 \cdot 10^{-34}}{mv}$.
   1. $\frac{6.626\times10^{-34}}{420\times1000/3600} = 5.679 \times 10^{-36} m.$
   2. $\frac{6.626\times10^{-34}}{10^{-3}\times10} = 6.626 \times 10^{-32} m.$
   3. $\frac{6.626\times10^{-34}}{10^{-9}\times1} = 6.626 \times 10^{-25} m.$

   Notice how small, even in the last case!

2. Using de Broglie's relation
$$p = \hbar/\lambda,$$
we find
$$p = \hbar|\boldsymbol{k}|.$$
The other of de Broglie's relations can be used to give
$$E = h\nu = \hbar\omega.$$
One of the important goals of quantum mechanics is to generalise classical mechanics. We shall attempt to generalise the relation between momenta and energy,
$$E = \frac{1}{2}m\boldsymbol{v}^2 = \frac{\boldsymbol{p}^2}{2m}$$
to the quantum realm. Notice that
$$\begin{aligned} \boldsymbol{p}\psi(\boldsymbol{r},t) &= \hbar\boldsymbol{k}\psi(\boldsymbol{r},t) = \frac{\hbar}{i}(\frac{\partial}{\partial x}\psi(\boldsymbol{r},t), \frac{\partial}{\partial y}\psi(\boldsymbol{r},t), \frac{\partial}{\partial z}\psi(\boldsymbol{r},t)) \\ E\psi(\boldsymbol{r},t) &= \hbar\omega\psi(\boldsymbol{r},t) = \frac{\hbar i\partial}{\partial t}\psi(\boldsymbol{r},t) \end{aligned}$$
Using this we can guess a wave equation of the form
$$\frac{-\hbar^2}{2m}\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2\right)\psi(\boldsymbol{r},t) = \frac{\hbar i\partial}{\partial t}\psi(\boldsymbol{r},t).$$
The momentum is represented by the three derivatives $\left(\frac{\hbar}{i}\frac{\partial}{\partial x}, \frac{\hbar}{i}\frac{\partial}{\partial y}, \frac{\hbar}{i}\frac{\partial}{\partial z}\right)$.

3. If $\psi_1(x,t)$ and $\psi_2(x,t)$ are both solutions of the time-dependent Schrödinger equation,
$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_1(x,t) + V(x)\psi_1(x,t) &= i\hbar\frac{\partial}{\partial t}\psi_1(x,t), \\ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_2(x,t) + V(x)\psi_2(x,t) &= i\hbar\frac{\partial}{\partial t}\psi_2(x,t). \end{aligned}$$
Add the right-hand sides and the left-hand sides, and use the sum rule for derivatives (e.g.,
$$\frac{\partial^2}{\partial x^2}\psi_1(x,t) + \frac{\partial^2}{\partial x^2}\psi_2(x,t) = \frac{\partial^2}{\partial x^2}\left(\psi_1(x,t) + \psi_2(x,t)\right).$$
This completes the proof!

4. The norm is defined as
$$\begin{aligned} \int_{-\infty}^{\infty} |\phi(x)|^2\,dx &= \int_0^l x^2(l-x)^2\,dx \\ &= \int_0^l l^2x^2\,dx - 2\int_0^l lx^3\,dx + \int_0^l x^4\,dx \\ &= \frac{1}{3}l^5 - 2\frac{1}{4}l^5 + \frac{1}{5}l^5 = \frac{l^5}{30}. \end{aligned}$$
The normalised form of $\phi(x)$ is thus
$$\begin{aligned} \phi(x) &= \sqrt{\frac{30}{l^5}}x(l-x) \quad 0 \le x \le l, \\ \phi(x) &= 0 \quad x \le 0 \text{ and } x \ge l. \end{aligned}$$

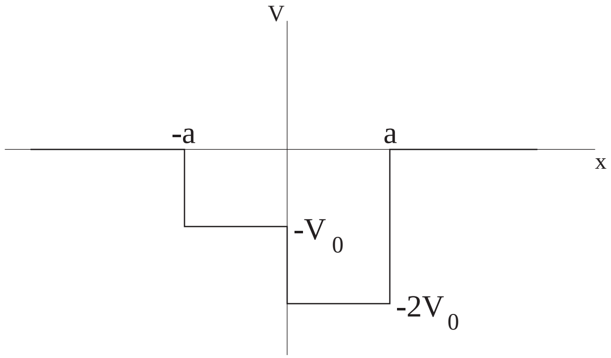

Figure 1: The potential

5. From left to right I define the regions I, II, III and IV. The continuity conditions take the form

$$\phi_I(-a) = \phi_{II}(-a),\;\; \phi_{II}(0) = \phi_{III}(0),\;\; \phi_{III}(a) = \phi_{IV}(a), \phi'_I(-a) = \phi'_{II}(-a),\;\; \phi'_{II}(0) = \phi'_{III}(0),\;\; \phi'_{III}(a) = \phi'_{IV}(a).$$

This answers the question, but for those of you who want to see more detail: We have to consider two cases,

- A $-2V_0 \leq E leq V_0$. we define

$$k = \sqrt{\frac{-2m}{\hbar^2}E},\;\; \kappa = \sqrt{\frac{2m}{\hbar^2}(E+2V_0)},\;\; k' = \sqrt{\frac{-2m}{\hbar^2}(E+V_0)}.$$

We then find

$$\begin{aligned} \phi_I(x) &= A_1 e^{kx} + B_1 e^{-kx} & \phi_{II}(x) &= A_2 e^{k'x} + B_2 e^{-k'x} \\ \phi_{III}(x) &= A_3 cos(\kappa x) + B_3 \sin(\kappa x) & \phi_{IV}(x) &= A_4 e^{kx} + B_4 e^{-kx}. \end{aligned}$$

(B_1 and A_4 are zero for reasons of normalisability.) This allows us to write explicit matching conditions.

- A $-V_0 \leq E leq 0$. The only function tat changes is ϕ_{II}. With

$$\kappa' = \sqrt{\frac{2m}{\hbar^2}(E+V_0)},$$

we find

$$\phi_{II}(x) = A_2 cos(\kappa' x) + B_2 \sin(\kappa' x).$$

6. As shown in the lectures, the wave function in region II is ether a sine or cosine, which means that in that region ϕ is even or odd. This implies $\phi(a)$ equals plus or minus $\phi(-a)$. Since the wave functions in regions I and III are identical exponents, we find that the whole wave function is symmetric or antisymmetric.

7. The wave functions are

$$\phi_l(x) = \begin{cases} \cos\frac{l\pi}{2a}x & l \text{ odd} \\ \sin\frac{l\pi}{2a}x & l \text{ even} \end{cases}$$

for $-a \leq x \leq 0$, 0 elsewhere.

For orthogonality we need to integrate two different solutions. One example, for $l \neq l'$, (why do I only integrate from $-a$ to a?)

$$\begin{aligned} &\int_{-a}^{a} \sin\frac{l\pi}{2a}\sin\frac{l'\pi}{2a}\,dx \\ &= \frac{1}{2}\int_{-a}^{a}\left[-\cos\frac{(l+l')\pi}{2a}x + \cos\frac{(l-l')\pi}{2a}x\right]dx \\ &= \frac{1}{2}\left[-\frac{2a}{(l+l')\pi}\sin\frac{(l+l')\pi}{2a}x + \frac{2a}{(l-l')\pi}\sin\frac{(l-l')\pi}{2a}x\right|_{-a}^{a} \\ &= 0 \end{aligned}$$

since $l+l'$ is even, and $\sin n\pi = 0$ for n integer.

The other two cases (cosine with cosine, and sine with sine) are done in a similar manner.

8. If $E < -V_0$, we define

$$\kappa = \sqrt{\frac{-2m}{\hbar^2}E}, \qquad \kappa' = \sqrt{\frac{-2m}{\hbar^2}(E+V_0)}.$$

In regions I and III the wave function is

$$\phi_{I,III}(x) = Ae^{-\kappa x} + Be^{\kappa x}.$$

Normalisation implies that

$$\phi_I(x) = B_1 e^{\kappa x}, \qquad \phi_{III}(x) = A_3 e^{-\kappa x}.$$

In the same way

$$\phi_{II}(x) = A_2 e^{-\kappa' x} + B_2 e^{\kappa' x}$$

If we match the wave function and its derivatives at $x = -a$ and $x = a$, we find

$$\begin{aligned} B_1 e^{-\kappa a} &= A_2 e^{-\kappa' a} + B_2 e^{\kappa' a} \\ \kappa B_1 e^{-\kappa a} &= \kappa' \left(A_2 e^{-\kappa' a} - B_2 e^{\kappa' a} \right) \\ A_3 e^{-\kappa a} &= A_2 e^{\kappa' a} + B_2 e^{-\kappa' a} \\ \kappa A_3 e^{-\kappa a} &= \kappa' \left(A_2 e^{\kappa' a} - B_2 e^{-\kappa' a} \right) \end{aligned}$$

Take the ration of the r.h.s. and l.h.s. of the first two equations, and of the last two, and find

$$\begin{aligned} (\kappa - \kappa')e^{-\kappa' a} A_2 + (\kappa + \kappa')e^{\kappa' a} B_2 &= 0 \\ (\kappa + \kappa')e^{\kappa' a} A_2 + (\kappa - \kappa')e^{-\kappa' a} B_2 &= 0 \end{aligned}$$

These equations can be shown to have only the zero solution, since the determinant,

$$(\kappa - \kappa')^2 e^{-2\kappa' a} - (\kappa + \kappa')^2 e^{2\kappa' a}$$

is negative for all values of κ'. This can be shown from the facts that $e^{4\kappa a} > 1$, and $(\kappa - \kappa')^2/(\kappa + \kappa')^2 < 1$.

9. Main similarities: alternating even and odd states. Main differences: for infinite well no continuity at boundaries, infinite number of bound states, no wave function outside the well.

10. Incoming and reflected wave in region I, outgoing wave in III, general solution in region II. Define

$$k = \sqrt{\frac{2m}{\hbar^2} E}, \qquad k' = \sqrt{\frac{2m}{\hbar^2}(E + V_0)}.$$

The wave functions are

$$\begin{aligned} \phi_I(x) &= A_1 e^{ikx} + B_1 e^{-ikx} \\ \phi_{II}(x) &= A_2 e^{ik'x} + B_2 e^{-ik'x} \\ \phi_{III}(x) &= A_3 e^{ikx} \end{aligned}$$

We match at a and $-a$,

$$\begin{aligned} A_1 e^{-ika} + B_1 e^{ika} &= A_2 e^{-ik'a} + B_2 e^{ik'a}, \\ ik(A_1 e^{-ika} - B_1 e^{ika}) &= ik'(A_2 e^{-ik'a} - B_2 e^{ik'a}), \\ A_3 e^{ika} &= A_2 e^{ik'a} + B_2 e^{-ik'a}, \\ ikA_3 e^{ika} &= ik'(A_2 e^{ik'a} + B_2 e^{-ik'a}). \end{aligned}$$

Multiply the last equation by $-1/(ik)$ and add to the penultimate one,

$$A_2 \left(1 - \frac{k}{k'} \right) e^{ik'a} = -B_2 \left(\frac{k}{k'} + 1 \right).$$

This can be solved for B_2,

$$B_2 = \frac{k' - k}{k' + k} e^{2ik'a} A_2.$$

Similarly we find

$$A_3 = \frac{2k'}{k + k'} e^{i(k' - k)a} A_2.$$

We can eliminate B_1 from the first two equations, multiplying the second with $1/(ik)$ and adding, using the relation for B_2 in terms of

A_2 found above,

$$\begin{aligned} A_1 e^{-ika} &= A_2 e^{-ik'a}\left(1+\frac{k'}{k}\right)+B_2\left(1-\frac{k'}{k}\right)e^{ik'a} \\ &= A_2 e^{-ik'a}\left(1+\frac{k'}{k}\right)+A_2\frac{k'-k}{k'+k}e^{2ik'a}\left(1-\frac{k'}{k}\right)e^{ik'a} \\ &= A_2\frac{e^{ik'a}}{k(k'+k)}\left((k+k')^2e^{-2ik'a}-(k'-k)^2e^{2ik'a}\right) \end{aligned}$$

This allows us to relate A_3 to A_1,

$$A_3 = e^{-2i(k-k')a}\frac{2kk'}{(k+k')}\frac{A_1}{2(k^2+k'^2)\cos 2k'a+4kk'\sin 2k'a}.$$

Since the wave numbers k in regions III and I are the same, the transmission coefficient T is

$$T=\left|\frac{A_3}{A_1}\right|^2=\frac{k'^2k^2}{(k^2+k'^2)\left[(k^2+k'^2)^2\cos^2 2k'a+4k'62k^2\sin^2 2k'a\right]}$$

The reflection coefficient R can either be calculated using similar means, or through the relation $R=1-T$!

11. We start from

$$\int_{-\infty}^{\infty} e^{-ay^2}dy=\sqrt{\pi}/\sqrt{a}.$$

Use

$$(-1)^n\frac{d^n}{da^n}\int_{-\infty}^{\infty}e^{-ay^2}dy\bigg|_{a=1}=\int_{-\infty}^{\infty}y^{2n}e^{-y^2}dy.$$

Differentiation the r.h.s. in the same way gives

$$(-1)^n\frac{d^n}{da^n}\sqrt{\pi}/\sqrt{a}\bigg|_{a=1}=(-1)^n\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\dots\left(-\tfrac{(2n-1)}{2}\right)\sqrt{\pi}=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

The integral with an odd power of y is zero, since the integrand has equal positive and negative contributions. We thus immediately conclude that

$$\int_{-\infty}^{\infty}\phi_l(y)\phi_{l'}(y)dy=0$$

if $l+l'$ is odd. The remaining integrals are

$$\int_{-\infty}^{\infty}\phi_0(y)\phi_2(y)dy=\int_{-\infty}^{\infty}(1-2y^2)e^{-y^2}dy=\sqrt{\pi}-2\frac{1}{2}\sqrt{\pi}=0,$$

and

$$\int_{-\infty}^{\infty}\phi_1(y)\phi_3(y)dy=\int_{-\infty}^{\infty}(y^2-\frac{2}{3}y^4)e^{-y^2}dy=\frac{1}{2}\sqrt{\pi}-\tfrac{2}{3}\tfrac{3}{4}\sqrt{\pi}=0.$$

12. First of all remember that a sum of solutions to the time dependent Schrödinger equation is a solution to that equation (see previous sheet).

The solution is

$$\psi(x,t)=\phi_1(x)e^{-iE_1/\hbar t}+\phi_2(x)e^{-iE_2/\hbar t}.$$

The probability is

$$|\psi(x,t)|^2=|\phi_1(x)|^2+|\phi_2(x)|^2+2\Re\left(\phi_1(x)\phi_2(x)^*e^{-i(E_1-E_2)/\hbar t}\right).$$

If ϕ_1 and ϕ_2 are real functions the time dependent part of this equation can be simplifies to

$$2\phi_1(x)\phi_2(x)\cos\omega_{12}t$$

with $\omega_{12}=(E_1-E_2)/\hbar$.

13.

$$\begin{aligned} \int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2}\,dx &= \left(\frac{\hbar}{m\omega}\right)^{1/2}\sqrt{\pi}, \\ \int_{-\infty}^{\infty}xe^{-\frac{m\omega}{\hbar}x^2}\,dx &= 0, \\ \int_{-\infty}^{\infty}xe^{-\frac{m\omega}{\hbar}x^2}\,dx &= ,\frac{1}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2}\sqrt{\pi}, \\ \int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2/2}\frac{\hbar}{i}\frac{d}{dx}e^{-\frac{m\omega}{\hbar}x^2/2}\,dx &= ,0 \\ \int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2/2}-\hbar^2\frac{d^2}{dx^2}e^{-\frac{m\omega}{\hbar}x^2/2}\,dx &= ,\frac{1}{2}\hbar m\omega\left(\frac{\hbar}{m\omega}\right)^{1/2}\sqrt{\pi}. \end{aligned}$$

Thus

$$\langle x^2\rangle = \frac{1}{2}\frac{\hbar}{m\omega}, \quad \langle p^2\rangle = \frac{1}{2}\hbar m\omega.$$

We conclude

$$\Delta x = \sqrt{\frac{1}{2}\frac{\hbar}{m\omega}}, \quad \Delta p = \sqrt{\frac{1}{2}\hbar m\omega}.$$

and

$$\Delta x \Delta p = \frac{1}{2}\hbar.$$

14. The worst the uncertainty in x can be is $2a$. For a constant function it is $\sqrt{\frac{2}{3}}a$, so a is not a bad estimate. From $\Delta p \approx \hbar/2\, 1/\Delta x$ we find $\Delta p \geq \frac{\hbar}{2a}$. Since the well is not moving, we expect the average momentum to be zero. We thus have

$$\langle \hat{p}^2\rangle \geq \frac{\hbar^2}{4a^2}.$$

The expectation value of the Hamiltonian is then

$$\langle H\rangle \geq \frac{\hbar^2}{8ma^2}.$$

Up to a factor π^2 this is the exact answer.

15. Assume $n < m$. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\left(\hat{a}^\dagger\right)^n e^{-y^2/2}\right]^* \left(\hat{a}^\dagger\right)^m e^{-y^2/2}\, dy \\
&= \int_{-\infty}^{\infty} \left[\left(\hat{a}\hat{a}^\dagger\right)^n e^{-y^2/2}\right]^* \left(\hat{a}^\dagger\right)^{m-1} e^{-y^2/2}\, dy \\
&= n\int_{-\infty}^{\infty} \left[\left(\hat{a}^\dagger\right)^{n-1} e^{-y^2/2}\right]^* \left(\hat{a}^\dagger\right)^{m-1} e^{-y^2/2}\, dy \\
&= nO_{n-1,m-1}.
\end{aligned}$$

Here we have used $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + \hat{1}$, and $\hat{a}e^{-y^2/2} = 0$. We thus find,

$$O_{n,m} = (n-1)...(0)O_{0,m-n} = 0.$$

16. The 1st and 2nd eigenstates of the infinite square well of width a are (normalised)

$$\begin{aligned}
\phi_1(x) &= \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}, \qquad & E_1 &= \frac{\hbar^2}{8ma^2}, \\
\phi_2(x) &= \frac{1}{\sqrt{a}}\sin\frac{\pi x}{a}, \qquad & E_1 &= \frac{\hbar^2}{2ma^2}.
\end{aligned}$$

We thus conclude that

$$\psi(x,0) = \frac{1}{\sqrt{5}}\phi_1(x) + \frac{2}{\sqrt{5}}\phi_2(x).$$

Thus we find

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{5}}\phi_1(x)e^{-iE_1t\hbar} + \frac{2}{\sqrt{5}}\phi_2(x)e^{-iE_2t\hbar} \\
&= \frac{1}{\sqrt{5}}\phi_1(x)\exp(-i\frac{\hbar}{8ma^2}t) + \frac{2}{\sqrt{5}}\phi_2(x)\exp(-i\frac{\hbar}{2ma^2}t).
\end{aligned}$$

We conclude that the possible outcomes of a measurement of the energy are E_1 with probability $1/5$ and E_2 with probability $4/5$.

17. Initially the wave function is $\phi(x) = \phi_1^0(x) = \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}$ for $|x| < a$, and zero everywhere else. After widening the well, the first three new eigenstates are (for $|x| < 2a$)

$$\begin{aligned}
\phi_1(x) &= \frac{1}{\sqrt{2a}}\cos\frac{\pi x}{4a} \\
\phi_2(x) &= \frac{1}{\sqrt{2a}}\sin\frac{\pi x}{2a} \\
\phi_3(x) &= \frac{1}{\sqrt{2a}}\cos\frac{3\pi x}{4a}
\end{aligned}$$

The probability of measuring the energy of each of these states is c_i^2, with

$$\begin{aligned}
c_1 &= \int_{-a}^{a} \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a} \frac{1}{\sqrt{2a}} \cos\frac{\pi x}{4a}\, dx \\
&= \frac{1}{\sqrt{2a}} \frac{1}{2} \int_{-a}^{a} \left(\cos\frac{\pi x}{4a} + \cos\frac{3\pi x}{4a} \right) dx \\
&= \frac{1}{\sqrt{2a}} \frac{1}{2} \left(\frac{4a}{\pi} \left[\sin\frac{\pi}{4} - \sin\frac{-\pi}{4} \right] + \frac{4a}{3\pi} \left[\sin\frac{3\pi}{4} - \sin\frac{-3\pi}{4} \right] \right) \\
&= \frac{1}{\sqrt{2a}} \frac{1}{2} \left(\frac{4a}{\pi} 2 + \frac{1}{2}\sqrt{2} \frac{4a}{3\pi} 2 \frac{1}{2}\sqrt{2} \right) \\
&= \frac{8}{3\pi}.
\end{aligned}$$

$$\begin{aligned}
c_2 &= \int_{-a}^{a} \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a} \frac{1}{\sqrt{2a}} \sin\frac{\pi x}{2a}\, dx \\
&= \frac{1}{\sqrt{2a}} \frac{1}{2} \int_{-a}^{a} \sin\frac{\pi x}{a} dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
c_3 &= \int_{-a}^{a} \frac{1}{\sqrt{a}} \cos\frac{\pi x}{2a} \frac{1}{\sqrt{2a}} \cos\frac{3\pi x}{4a}\, dx \\
&= \frac{1}{\sqrt{2a}} \frac{1}{2} \int_{-a}^{a} \left(\cos\frac{\pi x}{4a} + \cos\frac{5\pi x}{4a} \right) dx \\
&= \frac{1}{\sqrt{2a}} \left(\frac{4a}{\pi} \left[\sin\frac{\pi}{4} - \sin\frac{-\pi}{4} \right] + \frac{4a}{5\pi} \left[\sin\frac{5\pi}{4} - \sin\frac{-5\pi}{4} \right] \right) \\
&= \frac{1}{\sqrt{2a}} \left(\frac{4a}{\pi} 2 \frac{1}{2}\sqrt{2} - \frac{4a}{5\pi} 2 \frac{1}{2}\sqrt{2} \right) \\
&= \frac{8}{5\pi}
\end{aligned}$$

Probabilities: $\frac{64}{9\pi^2}$, 0, $\frac{64}{25\pi^2}$.

18. The Schrödinger equation is (for $l = 0$ $\phi(\boldsymbol{x}) = \phi(r)$, $\chi(r) = \frac{1}{r}\phi(r)$, $chi(0) = 0$)

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi_I(r) - V_0\chi_I(r) &= E\chi_I(r) && 0 \le r \le a \\
-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi_{II}(r) &= E\chi_{II}(r) && r > a
\end{aligned}$$

Define $k = \sqrt{\frac{-2mE}{\hbar^2}}$, $\kappa = \sqrt{\frac{-2m(V_0-E)}{\hbar^2}}$. We have

$$\chi_I(a) = \chi_{II}(a), \qquad \chi_I'(a) = \chi_{II}'(a).$$

We immediately find

$$\chi_{II}(r) = A_2 e^{-kr}$$

(no positive exponent due to normalisability) and

$$\chi_I(r) = B_1 sin\kappa r$$

(no cosine due to boundary condition at 0). Perform the matching at $r = a$,

$$\begin{aligned}
A_2 e^{-ka} &= B_1 \sin\kappa a, \\
-kA_2 e^{-ka} &= \kappa B_1 \cos\kappa a.
\end{aligned}$$

Take ratio of left and right hand sides:

$$\kappa a \cot\kappa a = -ka.$$

First state when $\sqrt{\frac{2m(V_0)}{\hbar^2}}a = \pi/2$.

19. $ka = \sqrt{\frac{2ma^2}{\hbar^2}E} = 0.462$ (use $2m = 1.66 \times 10^{-27}$kg, multiply MeV by $10^6 \times 1.602 \times 10^{-19}$ to obtain energy in J. From the hint we conclude $\kappa a = 1.80$. From the expression

$$\kappa a = \sqrt{\frac{2m}{\hbar^2}a^2 V_0 - (ka)^2}$$

we conclude

$$V_0 = \frac{\hbar^2}{2ma^2}\left[(\kappa a)^2 + (ka)^2\right] = (1 + (1.80/0.462)^2)2.33 = 37 \text{ MeV}.$$

Notice how much deeper than the binding energy!

20. Cartesian

$$\begin{aligned}\hat{\boldsymbol{L}} &= (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \\ &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\end{aligned}$$

Thus

$$\begin{aligned}&\hat{\boldsymbol{L}} z \exp(-\alpha(x^2+y^2+z^2)) \\ &= -i\hbar\left(y - 2\alpha z y^2 + 2\alpha y z^2, -2\alpha x z^2 - x + 2\alpha z x^2, 0\right)\exp(-\alpha(x^2+y^2+z^2))\end{aligned}$$

and

$$\hat{\boldsymbol{L}}^2 z \exp(-\alpha(x^2+y^2+z^2)) = \hbar^2 2 z \exp(-\alpha(x^2+y^2+z^2)).$$

Thus we have proven the answer.

Spherical: $\phi(r,\theta,\phi) = r\cos\theta e^{-\alpha r^2}$. With

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi},$$

and

$$\hat{\boldsymbol{L}}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi^2}\right)$$

The answer follows trivially.