Part 1 Particles and Forces

Session 2 Motion in the Plane

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Contents

Welcome

Welcome to session 1, part two of the Physics programme. In the previous session we studied the motion of point particles in one dimension. In this section we'll extend this to motion in more than one dimension, for example the motion of bodies orbiting under gravity. And we'll also look at how we can treat the motion of an extended body.

Our main problem in this session will involve an orbiting space station. This will raise further issues concerning gravitational forces and the dynamics of extended bodies. Getting to grips with this problem will cover most of the main topics in the unit. Then we'll look at some guided problems that revise much of the same material from various different angles.

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Learning Objectives

- Define angular speed and angular acceleration

- Explain the difference between centrifugal force and centripetal acceleration

- Derive the properties of circular orbits under various forces

- Show knowledge of Newton's laws for rotational motion

- Define moment of inertia and explain its use

- Define and use Torque

- Apply conservation of angular momentum

- Distinguish between stable and unstable equilibrium

- Define simple harmonic motion

The Problem

The problem in this session takes us into space. Let's go through the problem statement and unpick the learning issues. You might like to pause and consider what you think we need to find out to tackle this problem before continuing.

> **A design for a spaceship that would also function as an orbital space station might look like the dumbbell form of Spaceship _USS Discovery 1_ from the film _2001: A Space Odyssey._ The picture shows an artist's impression with the spaceship moving round the Earth oriented like a plane flying through the air. Is there anything wrong with this?**
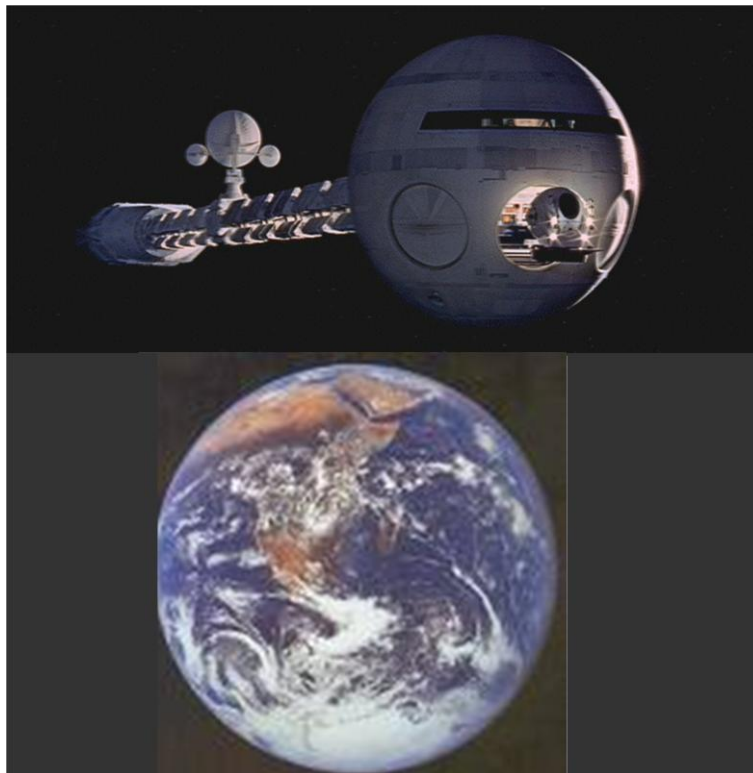
The first part of the problem seems to involve the orientation of the spacecraft. You may have had difficulty seeing what the author is getting at here. Why would the orientation matter in space? However, we're not in deep space in the picture: the spaceship is said to be moving round the Earth, so there's Earth's gravity to consider. The spaceship is in fact an orbiting satellite. What can we say about the orbits of satellites round a parent body? This is what we'll look at first.

The title of the second section gives a clue as to how we'll go on to answer the problem of orientation: we'll need to look at the concept of stability. Here we'll see a difference between the behaviour of a point mass and an extended body in orbit. We'll therefore have to consider how we model the mass distribution of the spaceship. Finally we'll look at the motion of the spaceship about its equilibrium position. This will take us into the dynamics of rotational motion and the important topic of simple harmonic motion.

How do we determine the orbit of a satellite? The problem doesn't explicitly ask for this, but we'll need it for the more difficult parts to follow, so this is where we'll start.

Circular Orbits under Gravity

Kinematics of circular motion: Angular Velocity

You may know that in general the orbit of a satellite round the Earth, or a planet round the Sun, is an ellipse. However, here we're going to restrict ourselves to circular orbits for simplicity. This will be sufficient to illustrate the main principles.

How do we describe motion in a circle? If you can, take a wheel, spin it round and think about what you see.

You might describe the motion in terms of the number of revolutions per unit time. This is the angular frequency of the wheel. Or you could say how fast various points on the wheel are moving, that is their linear speed. The important observation is that for a rigid wheel the points on a radius move together. This means we can describe their different linear speeds in terms of one angular speed – the angle turned through by a radius in unit time.
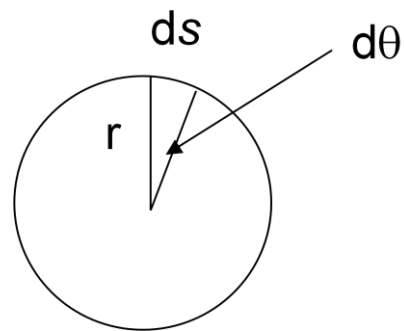
To get a mathematical statement of this, first recall the definition of an angle. Many students get so used to using angles that they forget the definition: measured in radians, an angle is the ratio of the length of the arc of a circle to its radius. So the small angle $d\theta$ in the figure is the arc ds divided by the radius r. Hence we get the first equation:

> ### r x dθ = ds
> **With θ measured in radians this is the *definition* of an angle!**

The speed of a point at radius r is the distance it travels through per unit time, hence ds/dt. Dividing equation (1) by a time interval dt we get the desired relation for speed:

> ### v = r dθ/dt

dθ/dt is the angular speed which we write equivalently as ω or $\dot{\theta}$. So we can express the key relation between linear and angular speeds as:

$$v = r\,\omega_r$$

in radians per unit time

What is the connection between frequency _f_ in cycles per second and angular speed ω in radians per second?

Kinematics of circular motion: Angular Acceleration

So much for speeds. To do dynamics we also need a description of the acceleration of a body moving uniformly in a circle.

Consider first the change in speed between two neighbouring points on the circle in figure 1. Because we're considering uniform motion round the circle, the speed is constant so the difference is zero. However, the acceleration isn't zero, because the *direction* of motion is changing. In order to remain in a circle the body trajectory is curved towards the centre. We can find the magnitude of the acceleration by comparing the two neighbouring *velocities*.
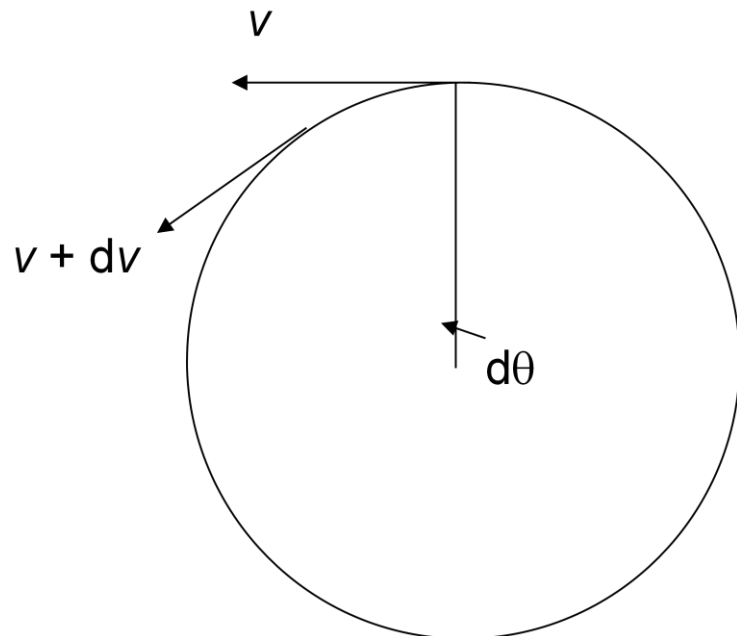
In the diagram, the velocity vectors from two neighbouring points have been taken from the circle and put into a triangle, so we can see clearly the change in velocity. The tips of the vectors lie on a circle because the lengths of both sides of the triangle represent the same speed v. This triangle therefore is exactly like the one on the previous page, where we compared two points on a circle.
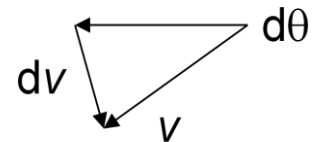
So just as on the previous page $ds = rd\theta$, so here $dv = v\,d\theta$. Dividing by dt, we get the acceleration dv/dt is equal to $vd\theta/dt$.

$$dv = vd\theta$$

$$dv/dt = vd\theta/dt$$
$$= v\omega$$
$$= v^2/r$$
$$= r\dot{\theta}^2$$

There are various equivalent forms obtained by using v=rω, some of which we've written down. The expression rω² for the acceleration is just what we might guess on dimensional grounds - rω² is the only way, up to a constant, that you can make an acceleration from a length r and an angular speed ω per unit time.

Note that the direction of this acceleration is towards the centre of the circle; we call this a centripetal acceleration.

Dynamics: Circular Orbits under Gravity

Turning now to the dynamics of circular orbits consider a body of mass M1 orbiting a central mass M2. To keep things simple we imagine that M2 is much more massive than M1 so we can neglect the influence of M1 on M2. Then the gravitational force between the masses produces an acceleration towards the centre of the orbit.
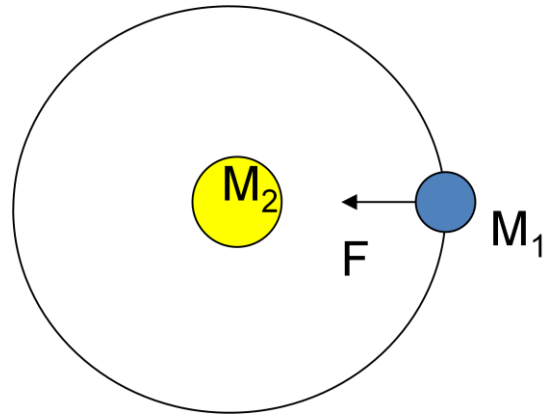
The force of gravity between two bodies of mass M1 and M2 a distance R apart is given by the inverse square law, equation (1). Big G in this equation is the gravitational constant, sometimes called Newton's constant. By Newton's third law, this force acts equally on both bodies, but as we've said, we're assuming the effect on M2 is negligible.

$$F_g = \frac{GM_1M_2}{R^2} \qquad (1)$$

$$g = \frac{GM_E}{R_E{}^2} \qquad (2)$$

The radius of the Earth can be determined by various means such as flying round it. The gravitational constant *G* can be measured experimentally to be 6.8×10^{-11} Nm^{2}kg^{-2}. is known. Use this, and the acceleration due to gravity, *g*, to estimate the mass of the Earth.

The acceleration of a body due to gravity is just the force divided by the mass of the body. So at the surface of the Earth the acceleration due to gravity is given by equation (2), where M_E is the mass of the Earth and R_E its radius. We can use equation (2) to estimate the mass of the Earth. Look up your answer on Wikipedia to see if you've done your sums correctly.

Compute the Orbit

We're now ready to compute the period of an orbit.

Consider the body of mass m_1 in the figure acted on by the gravity of the mass M at the centre of a circular orbit of radius r. The dynamics must express the fact that the orbiting body has just the right speed to be held in orbit by gravity of the central object.
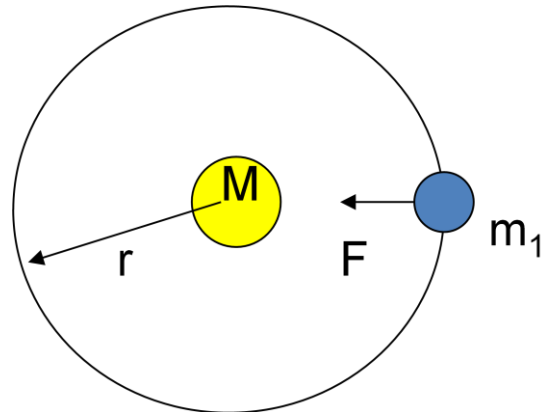
We can express this mathematically using F=ma, where F will be the gravitational force and a the centripetal acceleration. Using what we have learnt about these, gives us equation (1)

$$F = ma \rightarrow GM\, m_1/r^2 = m_1 r \omega^2 \qquad (1)$$

From this we can determine the period of an orbit of given radius in equation (2). Recall ω is in radians per sec so the period P is 2pi/ω.

$$P = 2\pi/\omega = 2\pi\, (r^3/GM)^{1/2} \qquad (2)$$

This result is in fact known as Kepler's third law – the period squared is proportional to the radius cubed. Note that the period is independent of the mass of the satellite.

Think about how you would explain why a more massive satellite doesn't orbit more slowly!

> **(Newton's Moon calculation) The Sun is 8 minutes away, the Moon 1.25 seconds. Using the ratio of the length of the month to the length of the year, estimate ratio of the masses of the Sun and the Earth**

You should find that the Sun is about 300 000 times more massive than the Earth.

Gravitional Potential Energy

You'll recall that the final part of our spaceship problem requires us to discuss energy. So to finish this section on orbits we'll look at the energy of an orbiting body.

The total energy will be made up of kinetic and potential energy. Returning to the equation of motion (1) of the orbiting body we can rewrite this in the form (2). The first term, mv^2 is just twice the kinetic energy. The second term must therefore be proportional to the potential energy. In fact the potential energy is exactly GM_1M_2/r.

$$\text{Equation of motion: } GM_1 M_2/r^2 = M_1 r\omega^2 = M_1 v^2/r$$

$$\text{or } M_1 v^2 - GM_1M_2/r = 0 \quad (2)$$

$$2KE + PE = 0 \quad (3)$$

We can check this by computing the gravitational force from the derivative of the potential energy. This gives us back the inverse square law so shows that we have identified the Potential energy correctly.

> **Why must the total energy of an orbiting body be negative? What would happen to a satellite if it were given a positive energy?**

Note that equation (3) relates the Kinetic and Potential energies for a circular orbit. The total energy E is the sum KE + PE which equals - KE, and must therefore be negative. This is as it should be – the orbiting bodies are bound together so energy must be put in to separate them to a large distance, where their total energy would be zero: thus their energy in orbit must be less than zero.

Summary

In this section we have learnt relations for the angular speed and angular acceleration. The force of gravity obeys Newton's inverse square law. The balance of force and angular acceleration gives us the orbit equation for a circular orbit. Finally we found an expression for the gravitational potential energy of two masses.

- Angular speed $\omega = v/r$

- Angular acceleration $= v^2/r = r\omega^2$

- Newton's inverse square law of gravity: $F = GM_1M_2/r^2$

- Circular orbits: $GM/r^2 = r\omega^2$

- Gravitational potential energy GM_1M_2/r

1. What is:

    (i)   the average angular speed

    (ii)   average acceleration of the Earth in its orbit about the Sun?

    (i) (a) 2π radians per year (b) 360^0 yr $^{-1}$ (c) 1 cycle per year (d) all of (a)-(c) (e) none of these

    (ii) (a) 9.81 ms^{-2} (b) 1 AU yr $^{-2}$ (c) 0 (d) none of these

2. If the angular velocity is ω what how many orbits are completed per unit of time?

    (a) $2\pi/\omega$ (b) $1/\omega$ (c) ω (d) $\omega/2\pi$

3. A satellite in orbit has (a) $|PE| > KE$ (b) $|PE| < KE$ (c) $|PE| = KE$

4. Put the following satellites in order of distance from Jupiter given their orbital periods in hours.

Mimas 22.62	Tethys 45.31	Titan 382.69
Enceladus 32.89	Rhea 108.42	Dione 65.69

The answers appear on the following page

Answers

1. (a), (b), (c) (d) In fact all of these answers (apart from (e)) are correct - this is the angle turned through in one year.
(e) Incorrect: you may be thinking that the speed varies, but it must average out to one orbit per year. Or you may have found the units confusing: they are not SI units, but they are perfectly valid.

2. (a) Wrong: The period = the time per orbit is $2\pi/\omega$ so the orbits per unit time is $1/(2\pi/\omega) = \omega/2\pi$.

   (b) Wrong: this is dimensionally wrong. We want the number per unit time, whereas $1/\omega$ has dimensions of time.

   (c) Wrong: This would be correct if ω were a frequency (cycles per second)

   (d) Correct . $\omega/2\pi$, if 2π is measured in radians.

3. (a) Correct: either from $2KE + PE = 0$ means that the PE is twice the KE in magnitude for a circular orbit (and therefore for any orbit close enough to a circle), or form $KE + PE < 0$ for a bound orbit.

   (b) Wrong: The PE is negative, so numerically less than the positive KE but the absolute value $|PE|$ is positive and greater than the KE.

   (c) Wrong: You may be confusing this with $2KE + PE = 0$ for a *circular* orbit.

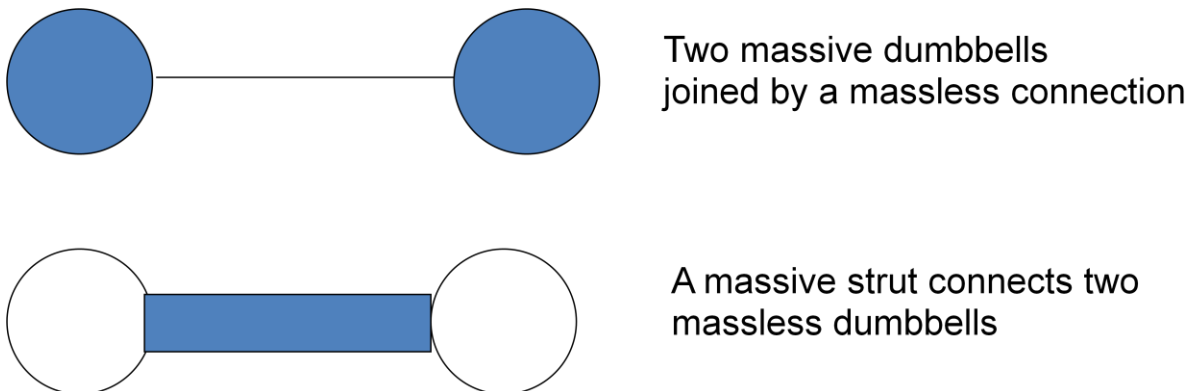4. The correct order is in increasing order of orbital period.

Equilibrium and Stability

So far we've seen how to calculate the properties of a circular orbit for a point mass. In this section we'll look at extended bodies.

Two ways of modeling the problem

We are now in a position to address the first of our problems, the orientation of the spacecraft.

To do this we cannot treat the body in orbit as a point mass. So we need to make some sort of model (or approximation) to the shape of the body. Clearly we want to keep it simple and not try to represent the detailed structure of USS 1. The diagram below shows two possible models:

Two massive dumbbells joined by a massless connection

A massive strut connects two massless dumbbells

We can make the dumbbells equal in mass and neglect mass of the joining section – or neglect the mass of the dumbbells and consider the mass to reside in the linear section

It doesn't matter which model we use to understand the problem and illustrate the principles involved. Certainly we would not want to use anything more complicated such as a massive strut connecting massive dumbbells!

We'll restrict ourselves to the top model in the following discussion, but you may like to consider, as we go through, what difference it would make to use the alternative model.

So we can return now to the problem of the orientation of the space craft. The figure shows two possible orientations. The symmetry in each case means there can be no net force or torque on the spacecraft, so in both of these pictures the spacecraft is in equilibrium. So where's the problem?
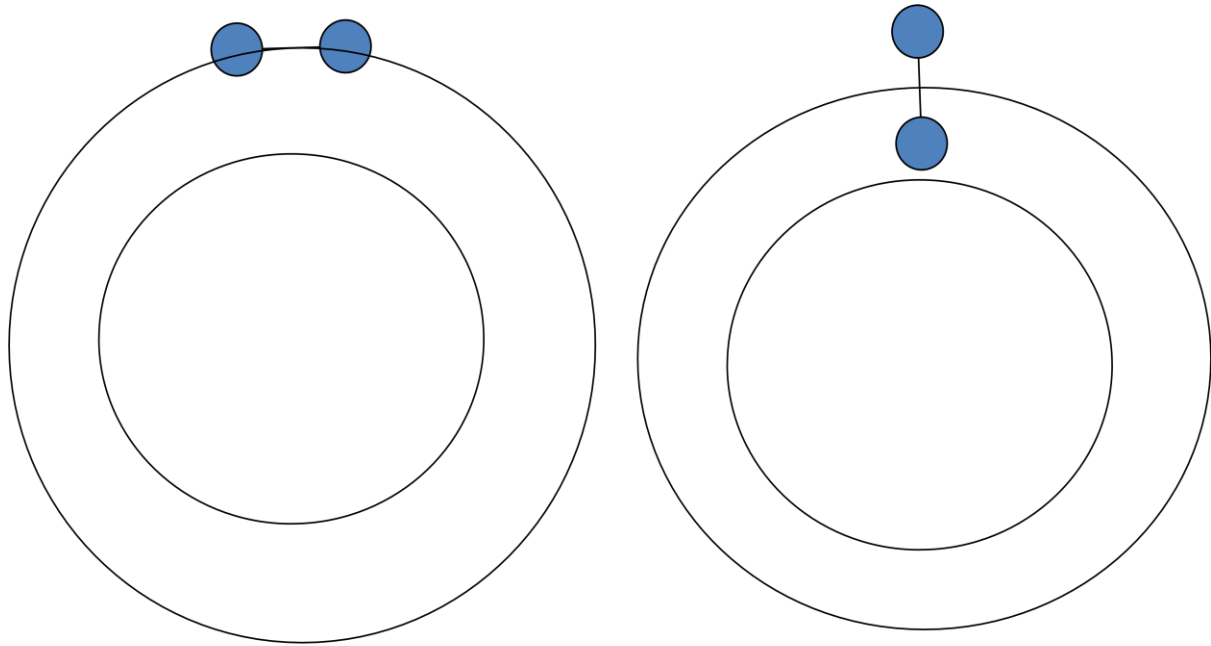
Two possible orientations: which one (if either) is stable?

Take your pencil and try to balance it on its point. You know in principle it should be possible. By the symmetry of the situation a balanced pencil would have no net force on it so should be in equilibrium. In practice what matters is not only the exact equilibrium point, but also what happens in the vicinity of that point. The smallest departure of the pencil from exact balance produces forces that tip it over. We say that such a situation is an unstable equilibrium. Conversely if a small displacement produces forces that tend to return the system to equilibrium, we say that the equilibrium is stable.

So which is it for the two orientations of the spacecraft? Which (if either) is like the pencil on its end, and which (if either) is like the pencil lying on the table? You might like to make a guess before going on.
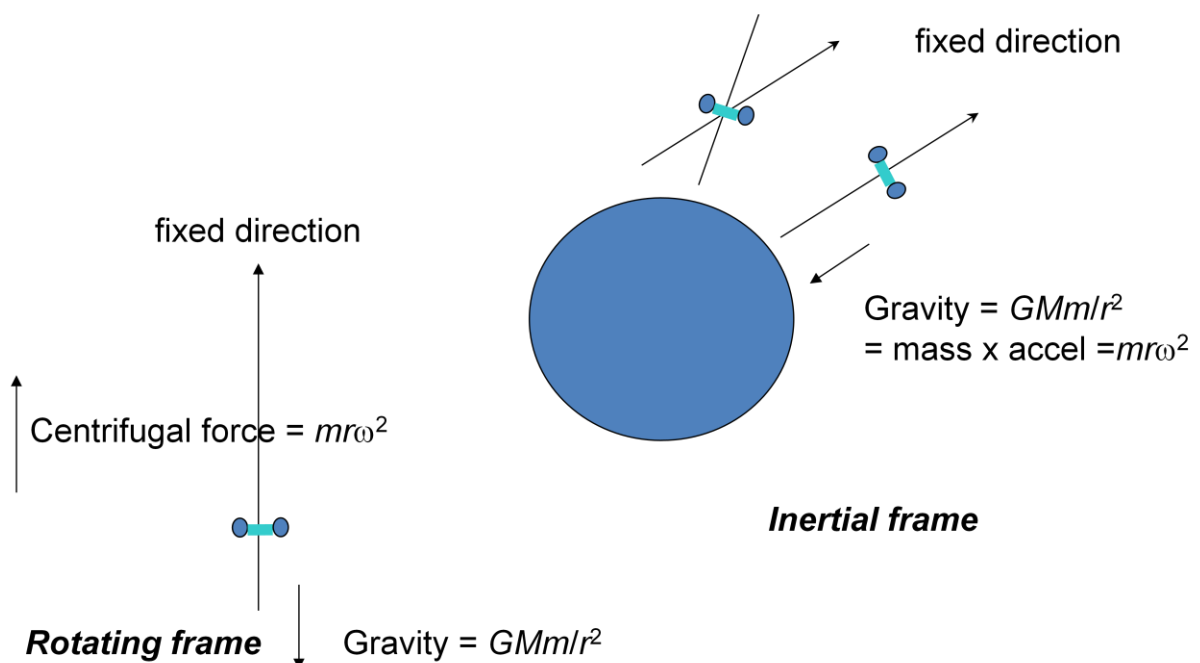
So far we've been looking at the spacecraft from the point of view of an orbit about a stationary Earth. That is to say, we've been using the Earth as our frame of reference. This is quite straightforward if the satellite can be described as a point mass, but for the motion of an extended body it's quite complicated. Putting ourselves in the frame of reference of the satellite appears to remove the orbital motion so the behaviour of the satellite can be described more simply. However, we must make the transformation to the rotating frame of the satellite correctly.

In a rotating frame of reference Newton's laws of motion no longer hold. We say that a rotating frame is not an inertial frame: in a rotating frame bodies apparently subject to no

forces appear to fly outwards! We therefore have to amend Newton's laws by introducing additional forces to account for this behaviour. An example, sufficient for our purpose here, is the centrifugal force. With the inclusion of centrifugal forces Newton's 2nd law is valid in a rotating reference frame.

Inertial frame:

  Newton's laws hold

  Force towards the centre produces a centripetal acceleration

Rotating frame:

  Newton's laws have to be *amended* by introduction of a centrifugal force

  Force towards the centre is balanced by outward centrifugal force

Orbits from the point of view of the rotating frame

fixed direction

Gravity $= GMm/r^2$
= mass x accel $=mr\omega^2$

Inertial frame

fixed direction

Centrifugal force $= mr\omega^2$

Rotating frame   Gravity $= GMm/r^2$

The diagram above shows why it is easier to consider the motion of the spacecraft in the rotating frame. Viewed from the Earth in an inertial frame the spacecraft rotates relative to a fixed direction as a result of its orbital motion. In the rotating frame the equilibrium

orientation remains fixed. We can therefore consider the motion about an equilibrium most easily in the rotating frame provided that we include the outward centrifugal force.

In the rotating frame the spaceship is at rest while the Earth spins about a fixed centre beneath it
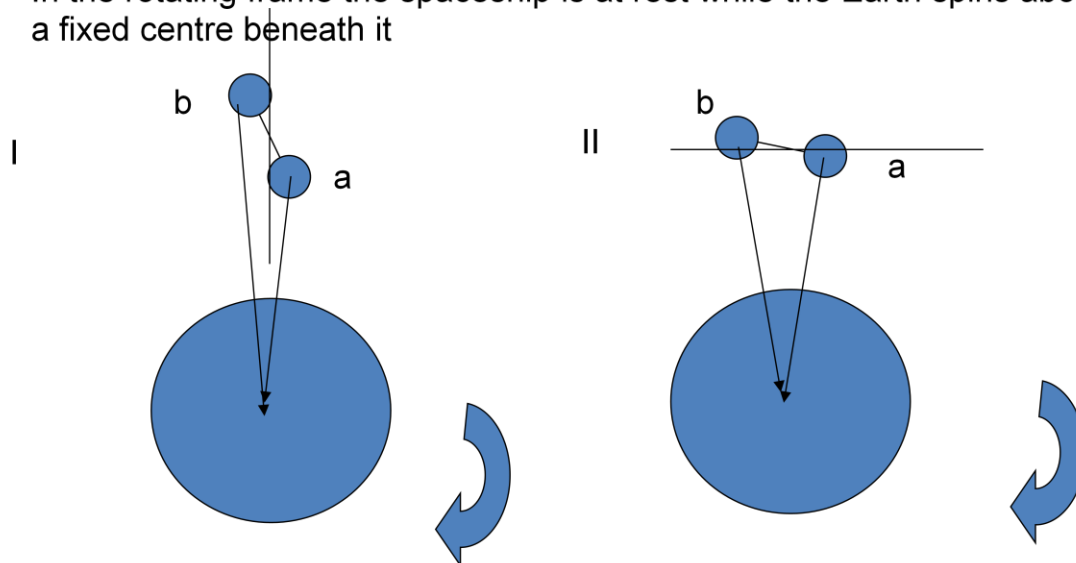
Figure I: the spaceship is slightly displaced from the vertical

Figure II: the spaceship is slightly displaced from the horizontal

Imagine then in this diagram that the satellite is at rest and the Earth is spinning beneath it. To answer the question of stability we imagine a small displacement of the satellite from equilibrium and look at the forces that then act on it.

In case 1 on the left, dumbell a is closer to the Earth than dumbell b. What does this mean for the gravitational force on each? Since the strength of gravity falls of with distance from the Earth – the $1/r^2$ law – the force on the closer dumbell (a) is greater. You can see that this creates a moment about the centre of mass that pulls the system back to the vertical. What about the outward centrifugal force? That on the dumbbell a is weaker than on b, which again acts to restore the system to equilibrium. So the equilibrium here is stable.

Using the same type of argument you should be able to explain why orientation II is unstable.

The result depends on the non-uniformity of the gravitational field – the fact that it changes with distance - which generates an imbalance of gravitational forces. It is a similar imbalance that generates the tides, so forces that arise from the non-uniformity of gravity are also called tidal forces.

Before we leave the subject of stability of equilibrium we'll look at a slightly different approach from the point of view of energy. This is useful because it provides a picture that translates into many other contexts. It's also often easier to use for calculations. That's
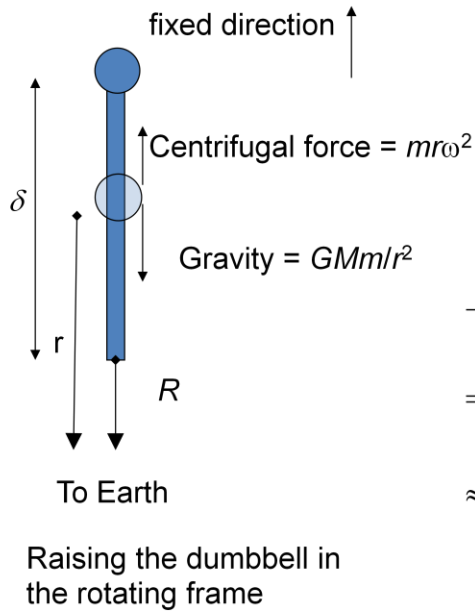
because in contrast to forces, energy is a scalar quantity so we don't have to worry about direction.
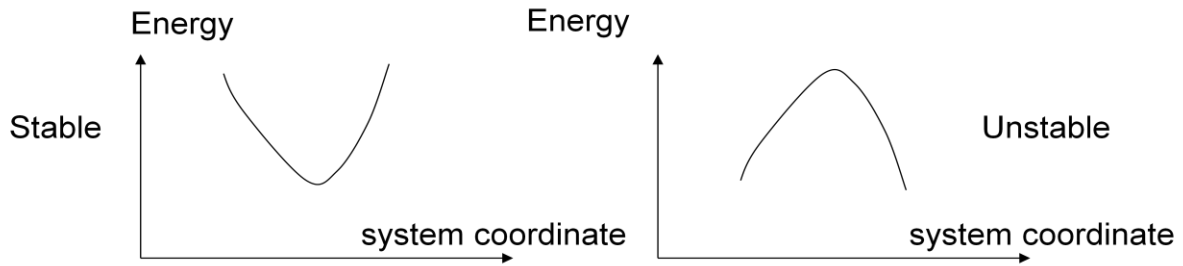
In this picture we plot the energy of the system against its configuration. In a general situation the configuration will be given by a set of parameters – various angles, lengths and so on – but for the diagram we'll imagine just one and plot it along the x-axis. We then plot the energy for various configurations along the y-axis. The picture is now quite intuitive. A system in equilibrium is in a minimum energy state. This is just like a ball rolling in a saucer. If the saucer is the right way up the ball rolls around and comes to rest at the centre. If it's the wrong way up the ball rolls off. Of course we can prevent this by imposing constraints – for example if we bluetac the ball to the upturned saucer. In the figure that would correspond to erecting little hills around the original peak.

Let's apply the energy picture to the satellite.

In the rotating frame the dumbbell is subject not only to the gravitational force but to the centrifugal force. Both have an associated potential energy.   For gravity we know the PE is GMm/r. For the centrifugal term we find the work done by the force $mr\omega^2$ in equation (1) in moving from the centre of mass of the dumbbell at R to the centre of the upper dumbbell at R + δ. This is the integral of the force over the distance as in equation (2). This is the loss in potential energy since we are moving in the direction of the force, which does the work. So the PE is – $\frac{1}{2}mr\omega^2$. The difference in potential energy between the Centre of Mass and the dumbbell centre is given in equation (3).

fixed direction ↑

Centrifugal force = $mr\omega^2$

δ

Gravity = GMm/r^2

r

R

To Earth

Raising the dumbbell in
the rotating frame

Net force at $r = -GMm/r^2 + mr\omega^2$ (1)

Gravitational PE = $-GMm/r$

Centrifugal PE = $-\int mr\omega^2 dr = -\tfrac{1}{2}mr^2\omega^2$ (2)

For the upper dumbbell, energy relative to R:

$$-\frac{GMm}{R+\delta} - \tfrac{1}{2}m\omega^2(R+\delta)^2 - \left(-\frac{GMm}{R} - \tfrac{1}{2}m\omega^2 R^2\right) \quad (3)$$

$$= -\frac{GMm}{R(1+\delta/R)} - \tfrac{1}{2}m\omega^2(R+\delta)^2 + \frac{GMm}{R} + \tfrac{1}{2}m\omega^2 R^2$$

$$\approx -\frac{GMm}{R}\left(1 - \frac{\delta}{R} + \frac{\delta^2}{R^2} + ...\right) - \tfrac{1}{2}m\omega^2(2R\delta + \delta^2) + \frac{GMm}{R}$$

$$= -\tfrac{3}{2}m\omega^2\delta^2$$

Total Potential Energy (for both dumbbells) = $3m\delta^2\omega^2$

To get any further we have to approximate this result for small delta. We can use the binomial expansion if you are familiar with that, or equivalently the sum of a geometric progression with common ratio δ/R. We keep just the first non-zero term in this approximation which is the term in δ^2.

We have to add on a similar quantity for the lower dumbbell – you can either go through the calculation again replacing $R+\delta$ by $R-\delta$, or simply replace δ by $-\delta$ in the final answer. This gives as a total potential energy of $3m\delta^2\omega^2$. Now let's see how this changes as the dumbbell is perturbed.

Imagine that the satellite is perturbed from its equilibrium position by an angle θ. We can see form the diagram that the radial distance of the dumbbell from the centre, δ, is $= d\cos\theta$. Thus we now have an expression for energy as a function of the system coordinate, θ. The figure shows $\cos^2\theta$ plotted against θ. We can see the minimum at $\theta = 0$ and the maximum at $\theta = \pi/2$. Thus orientation (a) is stable and (b) is unstable.

$$U(\theta) = -3m\omega^2 d^2 \cos^2 \theta$$

U(θ)

π/2

0   θ

(a)

(b)

m

δ=dcosθ

2*d*

θ

R_1

To Earth

R_2
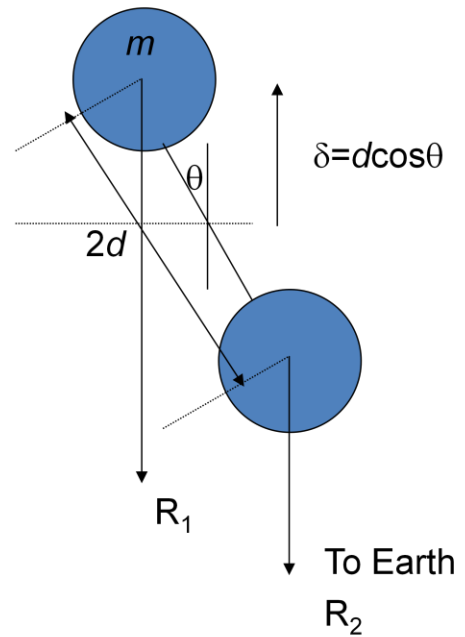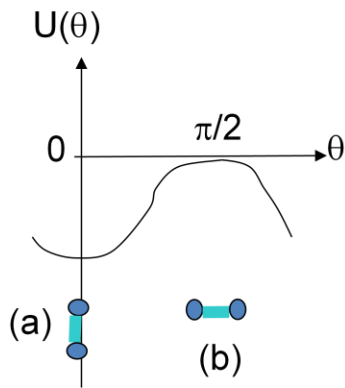
So θ=0 is at the bottom of a dip
in the PE curve, hence stable,
and θ=π/2 is at the top of a peak,
hence unstable.

**In a rotating frame centrifugal force balances the force
towards the centre**

**A stable system lies at a minimum of the energy, an
unstable one at a maximum**

SAQs

The Petronas Towers. Photo (cc) monstermunch99

1. The picture shows the Petronas Towers in Kuala Lumpur. The materials of the towers are clearly not in their lowest energy state. The building is obviously stable? Why?

   a) The energy picture doesn't apply to buildings.

   b) It's not stable to a bomb.

   c) It's stable to small perturbations in its energy well, even though another configuration has a lower absolute energy.

2. Applying the same logic to an airplane as to USS Discovery 1 would show that level flight is impossible and only a nosedive is stable. What is wrong with the argument in this case?

   a) The gravitational field at the Earth is effectively constant

   b) The controls on the airplane provide constraints to make it stable to small perturbations even though the nosedive has a lower absolute energy

   c) Both (a) and (b)

3. The actual (fictional) USS Discovery 1 had asymmetrical dumbells – what difference does this make to the stability argument?

    a) None

    b) The large mass will hang lower

    c) The smaller mass will hang lower

Answers

1. (a) Wrong: the energy picture always applies – it has to be drawn correctly though

    (b) This is true, but not the answer to the question. A large displacement may well take a system to a lower energy state (in fact bombs almost always do this). Stability has to do with the effect of small perturbations.

    (c) True: The energy diagram looks like this: DRAW FIGURE

2. (a) This is correct, but not the whole story: the differential forces here are due to the air flow.

    (b) Correct – it's essentially the same issue as in question 1. The controls make the plane stable to small perturbations. But tidal forces are not an issue here, so (a) is also relevant

    (c) correct – see the responses to (a) and (b) for an explanation.

3. (a) Correct. The centre of mass moves closer to the larger dumbbell which keeps the net moment exactly the same whatever the ratio of masses.   If you chose this answer on the grounds that the gravitational acceleration is independent of mass that is not correct: it's the force not the acceleration that matters here. The mass cancels when you compute the turning moment about the centre of mass.

    (b) Wrong: This would be the situation in a uniform gravitational field

    (c) Wrong: You might be thinking that centrifugal forces act more strongly on the larger mass. This is true when viewed in the rotating frame, but is accounted for in the change in centre of mass which means that the turning moment is the same.

Dynamics of rotational motion

Let's think a little bit more about unstable equilibrium. Think back to trying to balance your pencil on its point. The pencil falls over very quickly so the instability is very obvious. Suppose however that it took 100 years to fall over. We probably wouldn't care very much that such an equilibrium is unstable – for many purposes it might be considered as stable enough! There are many real life examples – buildings left to themselves for example do eventually fall down, rubber loses its elasticity and so on. So in many cases we would like to know not only if an equilibrium is unstable but the timescale for this to be apparent. Conversely, if an equilibrium is stable we might want to nevertheless see how the system behaves after a small displacement - a system might be stable in principle, but not before it has shaken itself to bits.

We'll investigate the general problem at the end of this section. First, as an example, we'll study the dynamical behaviour of our spacecraft. To begin with therefore we look at the motion of extended bodies.

In order to study the motion of the dumbbell when it is displaced from equilibrium, we need the equations of motion for a rotating body.

As we learnt in session 1, to get the equations of motion for a system, our starting point is always an expression for the energy.

Consider first a particle of mass m at radius r in a system rotating with angular speed ω about some fixed axis. Its kinetic energy is $\frac{1}{2}$ m v^2, which we can write as $\frac{1}{2}$ m r^2 ω^2. Why do this? Because ω is constant for the rotating body whereas v changes with distance from the rotation axis. Using ω rather than v we can sum up the energies of each of the particles that make up the extended body to get the total energy in the form of an expression involving a property of the body and ω.
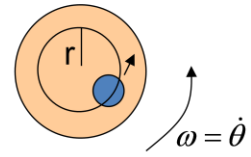
We see in equation (2) that the relevant property is called I, the moment of inertia about the axis of rotation. I is the sum over all mass points of (mr^2) where r is the distance from the axis.

Newton's Laws for rotational motion:

Kinetic Energy of a particle:

$$KE = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\omega^2 \qquad (1)$$

$\omega = \dot{\theta}$

For a body

$$\sum \tfrac{1}{2}mr^2\omega^2 = \tfrac{1}{2}\left(\sum mr^2\right)\omega^2 = \tfrac{1}{2}I\omega^2 \qquad (2)$$

Key Fact: KE of a rotating body = $\dfrac{1}{2}I\omega^2$

> **Look up the moment of inertia of a sphere and the moments of inertia of a cylinder of the same mass, about its long and short axes. Why are the moment of inertia different in the various cases?**

You can get *I* by integration, or looking up its value for simple shapes. For example, the moment of inertia of a uniform sphere of mass *M* and radius *R* is 2/5MR². Moments of inertia are related to the way in which mass is distributed throughout a body, specifically how close the mass is to the axis of rotation.

Next we need the potential energy. This will be some function of the angle of the system about the rotation axis. It's easiest to think of this in terms of a simple example. We could use the dumbbell satellite, but the PE we found for that was a bit complicated. A simpler example is the compound pendulum shown in the slide. Its potential energy at some angle θ is just the work done in raising the centre of mass from ½*l* to ½ *l* cosθ. This gives us equation (1)

Potential energy

$$PE = U(\theta)$$

Example: compound pendulum

$$U(\theta) = \tfrac{1}{2} Mgl(1 - \cos\theta) \quad (1)$$
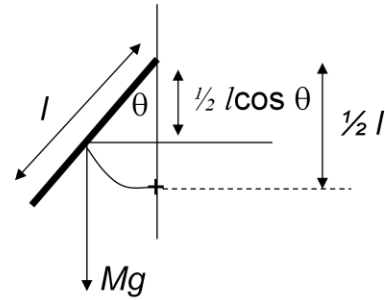
Equation of motion

Start from energy equation for a particle:

$$E = \tfrac{1}{2} I\dot{\theta}^2 + U(\theta) \quad (2)$$

Differentiate wrt time:

$$\frac{dE}{dt} = 0 = I\dot{\theta}\ddot{\theta} + \frac{dU}{d\theta}\dot{\theta} \quad (3)$$

or   (4)

where   is the torque on the system

Key fact: the equation of rotational motion for a body subject to a torque G is

$$G = I\ddot{\theta} \qquad (\text{Compare} \quad F = m\ddot{x} \ )$$

To derive the equation of motion we again start from an equation for the constant total energy of the system, equation (2). We then differentiate it with respect to time, as in equation (3). Recall that by the chain rule for derivatives, the rate of change of the potential energy U per unit time is the rate of change of U per unit angle times the rate of change of angle per unit time.

The derivative of the potential energy with respect to an angle gives us the moment of forces acting on a system which we call G. G is also called the couple or the torque. Thus dividing equation (3) by θ dot gives us our final result, equation (4). This says that couples produce angular accelerations in a way that is analogous to how forces producing linear accelerations.

We have defined the torque or couple acting on a system as the derivative of the potential energy with respect to angle.

You may already know that another way of looking at the torque or couple is as the moment of the forces about a point: G = rxF where *r* is the perpendicular distance from the point to the line of action of *F*. The direction of *G* is taken to be perpendicular to both the radius vector and the line of action of the force. The direction of G is then the axis about which the couple acts. In terms of vectors we say that the vector G is the vector product of the vectors r

and F: the magnitude of r times that of F times sin of the angle between them, in a direction perpendicular to both.

Key fact: G = r x F in a direction perpendicular to both r and F

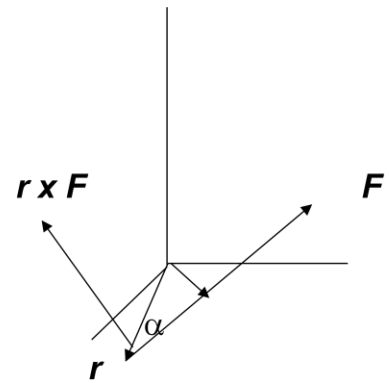**r x F**   **F**

α

r
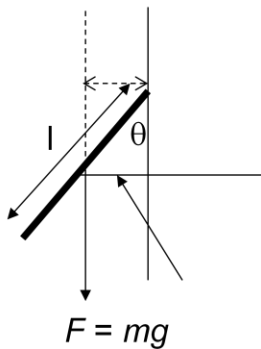
Example: compound pendulum

$$U(\theta) = \tfrac{1}{2} Mgl(1 - \cos\theta)$$

The magnitude of the couple is

$$\frac{dU}{d\theta} = Mg \times \tfrac{1}{2} l \sin\theta = r_\perp F$$

G = r x F
G = rFsin α

l

θ

F = mg

Its direction is obtained from the vector product or right hand rule

How can we reconcile this result with the definition of the couple as the derivative of the potential energy? We'll just take one example and show that the two are the same for this. Look at the compound pendulum again.

Differentiating the potential energy gives Mg times ½ l sin θ, that is the force times the perpendicular distance from the point of suspension to the line of action of the force, that is, exactly the moment of the force about the point of suspension.

We compared above the equations of motion for linear and rotational motion. They are repeated here on the top line of the table. We can also translate the laws of conservation of momentum and energy from the linear to the rotational context. The important point is that conservation of momentum in the absence of external forces becomes conservation of angular momentum in the absence of external torques. For a point mass m the conservation of angular momentum becomes I ω = $mr^2\omega$ = mrv = constant. This is an important result.

Equation of linear motion	Equation of rotational motion
$$F = m\ddot{x}$$	$$F = I\ddot{\theta}$$
Conservation of momentum $$Mv$$ if there are no external forces	Conservation of angular momentum $$I\omega$$ if there are no external torques
Conservation of energy of linear motion $$\tfrac{1}{2}mv^2$$	Conservation of energy of rotational motion $$\tfrac{1}{2}I\omega^2$$

Key fact: for a point mass the conservation of angular momentum reads

$$I\omega = mr^2\omega = mrv = \text{constant}$$

Using your thumb and forefinger to make a loop thread through some string with a small weight on the end holding the other end. Swing the weight in a steady motion. Now pay out extra length of rope trying to keep the swing constant. As you pay out more string the body orbits more slowly; as you draw in the string the body speeds up. How an this be explained?

Equations of motion for rotating bodies: $G = I\ddot{\theta}$
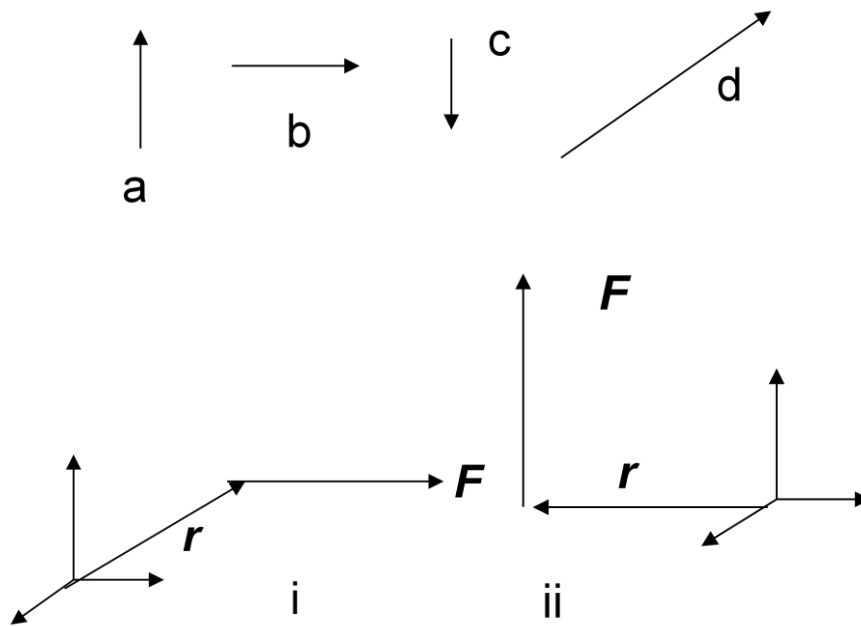
Moment of Inertia: $I = \sum mr^2$

External torque or couple: $G = r \times F$

Rotational Energy: $\frac{1}{2} I\omega^2$

Angular momentum: $I\omega$

1. A spinning ballet dancer draws in her arms thus halving her moment of inertia and doubling her angular velocity, so conserving angular momentum. Does her energy (a) increase, (b) decrease or (c) stay the same.

2. Match the couples to the forces:

Answers

1. (a) Correct – the energy comes from work done by drawing the arms in against centrifugal force.

   (b) You may be thinking that energy can only be dissipated, not created, which is true in a physical system without an internal energy source.

   (c) You may be thinking that the movements are internal to the system as a whole so cannot change the energy. But changing the angular momentum requires work and this comes from internal energy.

2. i matches c – the force is clearly acting clockwise viewed from above, so the couple points downwards according either to the rule from the vector product, or from the right hand screw rule.

   ii matches d. The force is creating a couple into the paper according to the right hand rule, or the vector product rule.

Simple Harmonic Motion

The motion of the dumbbell will be a small oscillation about its stable equilibrium position. The oscillation has a particularly universal form, called simple harmonic motion. Let's see how it works for the dumbbell.

In rotating frame the potential energy is

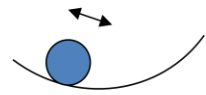$$U(\theta) = -3m\omega^2 d^2 \cos^2\theta \qquad (1)$$

The equation of motion is

$$I\ddot{\theta} = -\frac{dU}{d\theta} = -6m\omega^2 d^2 \cos\theta \sin\theta \qquad (2)$$

Small angle approximation:

$$\ddot{\theta} = -\frac{6m\omega^2 d^2}{I}\theta = -3\omega^2\theta \qquad (3)$$

since $I = 2md^2$

The period is approximately $2\pi/\sqrt{3}\omega = 2\pi(R^3/3GM)^{1/2}$

This is independent of the mass and length of the satellite.

We derived the potential energy of the dumbbell, equation (1) in the rotating frame in section 2. We found the general form for the equations of motion, I θ double dot = -dU/dθ in section 3. Putting these together gives us equation (2) where we've used the fact that the derivative of $\cos^2\theta$ is -2 sinθcosθ . To gte any further we assume that θ is small. Then cos θ is approximately 1 and sin θ is approximately θ. Also the moment of inertia if the two dumbbells, considered as point masses on the end of the link of length 2d, is $2md^2$. This gives us equation (3) which we can use to derive the period of oscillation.

The solution of equation (3) is θ equals sin (or cosine) of (ω√3 t), so the frequency of oscillation is just √3ω and the period is 2π/√3ω. Oscillations of the form cos or sin ωt are called simple harmonic and the system is said to undergo simple harmonic motion or shm. This behaviour arises from equations of motion of the form (3) in which the acceleration is

proportional to the negative of the displacement. The constant of proportionality in the shm equation of motion is the square of the oscillation frequency.

If a system displaced from equilibrium undergoes shm it is stable: in practice friction will eventually damp out the oscillations and the system will return to equilibrium. In terms of our energy diagram for stability, the equation describes a ball oscillating in the potential well.

What is the situation if we make a small perturbation from unstable equilibrium. The answer is that we would get an equation like (3) except with a plus sign on the right hand side. What does the period then correspond to? – clearly not an oscillation. In fact, it will be the time taken for the instability to grow by an exponential factor, essentially the time for the system to tip over.

Simple Harmonic Motion General Theory

Having investigated a particular example, we'll now look at general situation in which SHM occurs. We start as usual from the energy equation, assuming that the system has the usual kinetic energy and a potential energy that is a quadratic function of position. The quantity ω here is a general parameter not the same quantity that has appeared in the orbit equations previously.

Problem: If a body is perturbed from equilibrium, how does it move?

Recall equilibrium: approximate potential as quadratic near minimum

$$E = \tfrac{1}{2} m\dot{x}^2 + C + \tfrac{1}{2} m\omega^2 x^2 \qquad (1)$$

Equation of motion $\qquad \ddot{x} = -\omega^2 x \qquad (2)$

The motion is called SHM – restoring force prop to displacement

Problems in SHM amount to finding ω from the appropriate potential

The derivative of the energy equation with respect to time gives the equation of motion (2). Here we see shm again. It's the general result of a quadratic potential energy, which is itself the general result of a perturbation from a stable equilibrium. Thus, problems in shm amount to finding the quadratic part of the potential energy and identifying the frequency from this.

Summary

Simple Harmonic Motion: $\ddot{x} = -\omega^2 x$

Acceleration = - (frequency)^2 x displacement

Match the frequencies of SHM for a body of mass m near $x = 0$ to the potentials

a) $U = m\omega^2 x^2$

b) $U = m\cos(\omega x)$

given that $\cos(\omega x)$ can be approximated as $1 - \frac{1}{2}(\omega x)^2$ for small $\omega^2 x$

c) $\dfrac{m}{1 - 2\omega^2 x^2}$

(i) ω

(ii) $\sqrt{2}\omega$

(iii) 2ω

(iv) $\omega/2$

The answers appear on the following page

Answers

(a) = (ii)   (b) = (i)   (c) = (iii)

In all cases the frequency is found from the quadratic term by taking dU/dx. So $d/dx(m\omega^2 x^2)$ $= 2\, m\omega^2 x$ and the frequency is $\sqrt{2}\omega$. (c) is a bit more difficult: you need to recall the geometric series (or binomial expansion) to get $U = m(1 + 2\, \omega^2 x^2)$.

Summary of the spaceship problem

Angular speed ω is measured in radians per unit time

The acceleration in a circular orbit is v^2/r or $r\omega^2$.

The Potential energy in a circular orbit = $-\frac{1}{2}$ the kinetic energy

In a circular orbit, viewed from an inertial frame, gravity causes a centripetal acceleration. Viewed from the rotating frame gravity is balanced by a centrifugal force

A stable system is at the bottom of an energy well. When perturbed it executes simple harmonic motion

The equation of motion for a rotating system subject to a couple or torque G is $G = I$ where I is the moment of inertia = Σmr^2. The couple $\mathbf{G} = \mathbf{r} \times \mathbf{F.}$

In an isolated system angular momentum is conserved.

Additional Problems

Kepler's Law

Kepler's third law of planetary motion gives the relation between the period of a planet in its orbit and the radius of the orbit.

Derive such a relation from the formula for the period of the pendulum.

$L \rightarrow R$

$g \rightarrow const /R^2$

Therefore T^2 prop to $1/R^3$

Having got the period of a pendulum in session 1 let's make some use of it. A planet in orbit is a bit like a pendulum (on the end of a gravitational 'string'). The length is the radius of the orbit and $g = GM/R^2$ is the gravity of the Sun . So the period is $(R^3/GM)^{1/2}$ which is Kepler's third law.
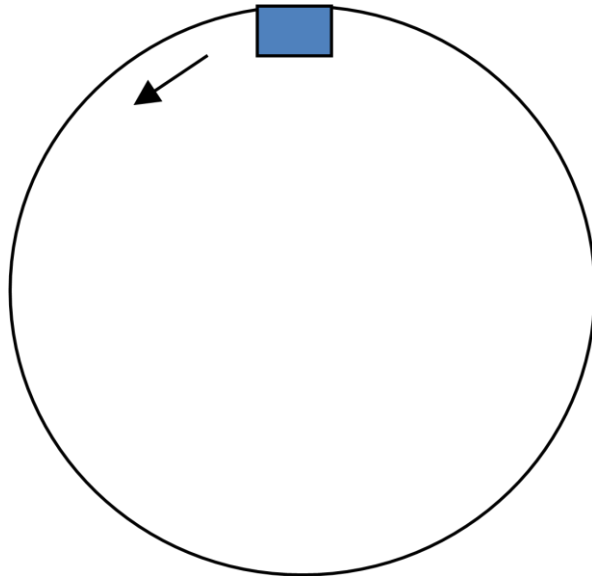
This shows the power of physics to illuminate connections.

Kinematics of circular motion

Assuming that the carts are not attached to the rails how big can a circular track in a fairground ride be?

The ride. Photo (cc) Bill Rhodes.

The Model

This problem doesn't seem to give you much to go on. Why would there be a limit to the size of a circular ride? Think about it in stages. There must be a condition somewhere for the ride to work. Fix a size in your mind and ask what problem there might be? – It's going to be where the cart is upside down. How will the passengers not fall out. Well, you could strap them in. Then how will the cart not fall off the rails. You could engineer this by shaping the rail, but the increased friction would be very energy intensive. Better to make sure the cart is going fast enough. How does the speed requirement go as the size gets bigger? Why can't it go as fast as you like at the top? Of course, you can. But what is the problem at the bottom? Speed isn't a problem, so what is?

Do the problem – need $v^2/r = g$ at top So $v^2 = gr$; then at bottom,

$V^2 = v^2 + 2g(2r)$ by conservation of energy, or $V^2/r = 5g$

It doesn't matter what the radius is!

Any restriction is caused by frictional losses.

Orbits

The Sun is losing mass constantly via the conversion of nuclear fuel and a solar wind. How does the loss of mass affect the orbit of the Earth? What impact does this have on climate change?

To tackle this problem we need to know how the orbit of a planet depends on the mass of the Sun. A simple model is to take the orbits to be circular. (You might like to think how we know that taking elliptical orbits wouldn't make much difference).
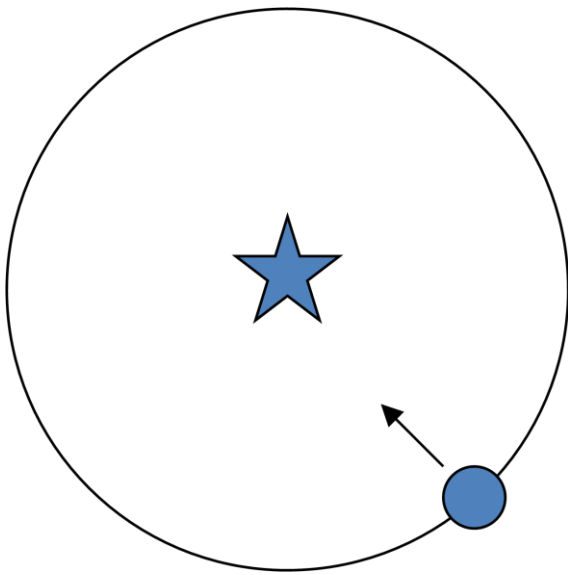
We determine the orbit as usual by balancing gravity and centripetal acceleration. This gives us the radius of the orbit as a function of the mass of the Sun if we know v. We can get another relation from conservation of angular momentum since there are no tangential forces. This lets us eliminate v to get that the orbital radius is inversely proportional to the mass. The mass loss from the Sun is unlikely to provide a significant amelioration of climate change.

In an inertial frame Force = mass x centripetal acceleration

$GMm/r^2 = mv^2/r$

How is v determined? – Conserve angular momentum

mvr = constant or $v \propto 1/r$

Hence Mr = constant

Galileo's inclined plane

Galileo is credited with establishing that all bodies fall with the same acceleration under gravity. He did this not by dropping bodies from the leaning tower of Pisa, as legend has it, but by rolling balls down an inclined plane. This has the advantage of diluting gravity which makes it easier to measure the time of fall. However, Galileo was fortunate in the shapes of bodies he chose to compare.

Does the shape of the rolling object make any difference?

Would Galileo have made his discovery if he had compared the rolling motion of spheres and cylinders?

We'll leave this one to you. It can be solved by writing down an expression for the energy of a rolling cylinder or sphere. This has three parts: the linear motion, $1/2mv^2$; the rolling motion $1/2I\omega^2$ or, in terms of v, $\frac{1}{2} I v^2/r^2$ and the potential energy, mgz. Differentiating with respect to time will show that the acceleration is constant, so you can use the constant acceleration formulae to get the time of decent. The answer will depend on *I* and hence on the shape.

Wall of Death

A Fairground rotor consists of a cylindrical room which can be spun about its axis of symmetry. Intrepid members of the public stand with their backs to the wall while the room is spun up, at which point the floor is removed. The people inside find themselves stuck to the wall. How fast must a 4m rotor be spinning before the floor can be lowered?

Photo (cc) Benny Mazur.

In the frame of the room the centrifugal force must be sufficient that the reaction from the wall created sufficient friction to hold up a bodyweight. Thus $mr\omega^2$ must be sufficiently greater than mg.

In order to remain in orbit a spacestation must be supplied with energy. An efficient way of doing this is to use the energy of visiting space shuttles as they undock, by paying out the shuttle on a long tether. How would this work?

The orbit of a satellite in near Earth orbit decays – why?

This means that energy has constantly to be supplied to keep the spacecraft in orbit.

On the other hand a visiting Shuttle craft has to loose energy before if can return to Earth. A tether payed out between them can transfer energy. How?

First, why must the spaceship be supplied with energy to remain in orbit? What does paying out a tether mean? How could energy be transferred from one spacecraft to the other without the use of power? How would this fit in with the conservation laws? Take a few moments to think about these questions.

The orbit of the spacestation can only decay if it loses energy. This implies some frictional force, and indeed in near Earth orbit the atmosphere is still dense enough to affect something like the international space station. Recall from session 1 that this creates a drag on the spacestation. So this explains why we need to transfer energy. To think about how to do so it's not necessary to worry about frictional losses, so we'll ignore these from now on.

Since there are then no dissipative forces acting on the system, the total energy must be conserved. Since there are no external couples acting on the system the angular momentum must be conserved. This gives us a clue as to what is proposed. If the shuttle is losing angular momentum as well as energy then that angular momentum and energy must be transferred to the spacestation. What affect will this have: one might guess that it will boost the spacestation to a higher orbit. Let's calculate it and see.

We give the shuttle a little nudge towards the Earth to start things off. We now have a system similar to the dumbbell in a radial orientation – a lower mass joined to a higher one. Except that now the connection is not rigid. Looked at In the rotating reference frame we know that there is a net downward force on the lower mass (here the shuttle) and an outward one on the upper mass, here the spacestation. Since the link between them is not rigid these forces will cause the tether to be payed out without any further expenditure of energy.
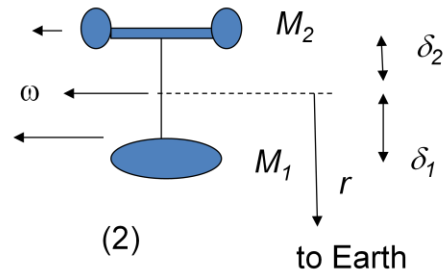
Energy: KE $= -\frac{1}{2}$ PE (section 1)   (1)

M_2

δ_2

ω

Total Angular Momentum = Shuttle + Spaceship

M_1 δ_1

r

$L \quad = \quad M_1(r + \delta_1)^2\omega + M_2(r + \delta_2)^2\omega$ (2)

to Earth

So $dL = 0 = 2M_1 r\,\delta_1\omega + 2M_2 r\,\delta_2\omega \Rightarrow \delta_1/\delta_2 = -M_2/M_1$ (3)

$0 = M_1\,\delta_1 + M_2\,\delta_2$ (4)   CM remains fixed

Shuttle is going too slowly for orbit – so speeds up as it loses energy.
This transfers angular momentum to the spaceship
Spaceship gains energy and slows down as it moves out.

As the shuttle falls its kinetic energy and potential energy change. We saw earlier that in a circular orbit the KE = -1/2PE. So half the PE it loses goes into KE. An increase in KE means that the shuttle speeds up. This is correct: bodies closer in go round faster. But where has the other half of the change in PE gone? The only possibility is into increasing the energy of the spaceship.

To see how this affects the orbit, consider the angular momentum of the system. This is made up of a contribution from the shuttle at a radius r+ δ_1 and from the spaceship at r+ δ_2 . This is equation (2). Note that the system moves with a constant angular speed to maintain a radial orientation. The change in angular momentum in this state from one in which the shuttle and spacestation were together is obtained by subtracting the original angular momentum. In equation (3) we've done this and also neglected the δ^2 terms, assuming that δ is small compared to r. This gives us the ratio of the distance moved by the spacestation compared tot hat moved by the shuttle. The negative sign shows that these are in opposite directions as we would expect.

Equation (4) is obtained by rewriting (3). It expresses the fact that the centre of mass of the system does not change as the spacecraft move apart, as we might expect, since there is no net energy change.

The net result is that the shuttle is slowed down as is required for re-entry and the energy, instead of being wasted goes into raising the orbit of the orbiting spaceship.

Summary

Vertical and horizontal components of motion are independent

Momentum is conserved in an isolated system

Energy is conserved in an elastic collision

Collision damage is caused by inelastic transfer of energy

In an inertial frame central forces produce centripetal acceleration

For motion in a circle v=rw, for centripetal acceleration = rw^2 = v^2/r, angular momentum = mvr

Kinetic energy of a rigid body is ½ Iw^2

Angular momentum is Iw

Equation of motion is G=Idw/dt

Tutorial Problems

1. In order to prepare for space missions astronauts are subject to high g-forces strapped into a centrifuge. A female of height (5′6″) can withstand about 9g whereas a 6′3″ male blacks out at 7g. Explain.


2. Where is the angular momentum in the Solar System? Don't forget you always have the option of transforming frames of reference. Try looking at it from the astronaut's point of view (at rest).

3. The Moon recedes from the Earth at the 3.8cm/year as a result of tidal torques exerted by the Earth on the Moon. How would you explain this? Estimate the change in the length of the year as a result of this.

4. Look up the maximum power and maximum torque and the corresponding revs (the angular speed of the engine) for your car (or for a car of your choice on the internet if you prefer)

Work out the torque of your car at maximum power and compare this with the figure given for maximum torque

Hint: Power = torque x angular speed
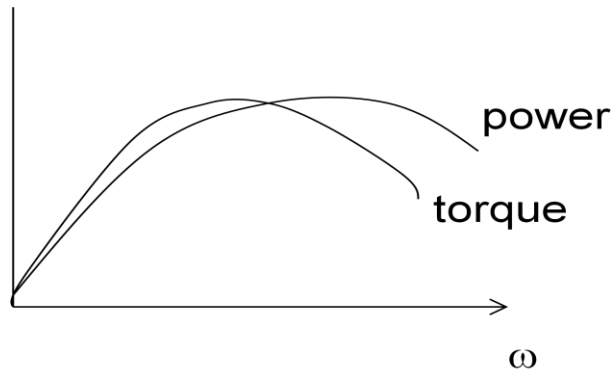
It's useful here to plot a graph of torque and power against revs. Because power is torque x revs it peaks at a higher omega. Torque gives acceleration, power gives you speed.

Meta tags

Author: Sarah Symons.

Owner: University of Leicester

Title: Enhancing Physics Knowledge for Teaching – Motion in the Plane

Keywords: Forces; Orbits; Stability; Simple Harmonic Motions; sfsoer; ukoer

Description: In this section we'll extend this to motion in more than one dimension, for example the motion of bodies orbiting under gravity. And we'll also look at how we can treat the motion of an extended body.

Language: English

Version: 1.0

Additional Information

This pack is the Version 1.0 release of the module. Additional information can be obtained by contacting the Centre for Interdisciplinary Science at the University of Leicester.
http://www.le.ac.uk/iscience