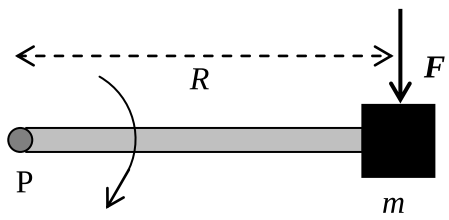
Section 7: Rotational Kinematics

7.1 Torque and Newton's second law

There are two conditions for static equilibrium of a body, namely that (i) the vector sum of all the external forces acting on the body must be zero, and (ii) the sum of all the external torques acting on the body must be zero. If the first of these conditions is not satisfied then, from Newton's second law, the body would experience a linear acceleration proportional to the magnitude of the net force. Similarly if the second condition is not satisfied we would expect the body to experience an *angular* acceleration proportional to the net torque.

Consider a mass m attached to a light rod, pivoted at the point P. If a force F is applied to the mass at right angles to rod, there will be an instantaneous linear acceleration a, where

$$F = ma$$



Since the mass is pivoted at P, it is constrained to move in an arc of radius R, so the linear acceleration a is converted into an angular acceleration α, such that $a = \alpha R$. The force equation therefore becomes:

$$F = mR\alpha$$

If we now multiply both sides by the radius R, the product FR on the left hand side becomes the torque Γ:

$$\Gamma = FR = mR^2\alpha = I\alpha$$

This is Newton's second law applied to rotational motion. $\Gamma = I\alpha$ has exactly the same form as $F = ma$, with the torque Γ replacing the force F, the angular acceleration α replacing the linear acceleration a and the role of the mass m being taken by I, *the moment of inertia*.

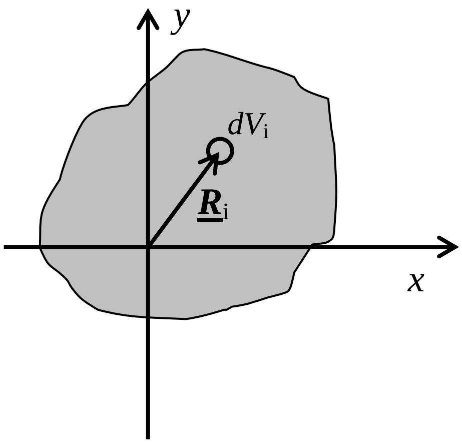
For a point mass rotating about a pivot at a radius R, the moment of inertia is defined as $I = mR^2$. We will see how to calculate the moment of inertia for extended objects below.

7.2 Moments of inertia

In the context of rotational motion, the moment of inertia plays a role equivalent to mass in linear kinematics. As we have seen above, the moment of inertia for a point mass m rotating about an axis at a radius R has the value $I = mR^2$. Suppose we have a composite system made up of several masses m_i, at various distances R_i from a common axis about which the system rotates at a common angular velocity. The total moment of inertia of the system is then the sum of the individual moments of inertia:

$$I = \sum_i m_i R_i^{\,2}$$

The moment of inertia of a rigid body



Suppose that a rigid body is rotating about the z axis and that the body has a uniform density ρ. We can divide the body up into a set of discrete volumes dV_i, so that the total mass m is given by

$$m = \sum_i dm_i = \sum_i \rho \, dV_i$$

We can now calculate the moment of inertia of the body by summing over all the mass elements $\rho \, dV_i$, weighted by the square of the distance of each element from the rotation axis:

$$I = \sum_i R_i^2 \, dm_i = \sum_i R_i^2 \rho \, dV_i$$

When the volume elements dV_i are infinitesimal this can be written as an integral:

$$I = \int R^2 dm = \int R^2 \rho \, dV$$

where the integral is taken over the volume of the body.

A uniform circular disc

The moment of inertia of a uniform disc, rotating around an axis passing through its centre and perpendicular to it, can be easily by integration, but an alternative method of calculation is given here which avoids calculus. Firstly, notice that the moment of inertia I must be proportional to the disc's mass M (if its radius is fixed) and to the square of its radius R (if its mass is fixed). That is to say,

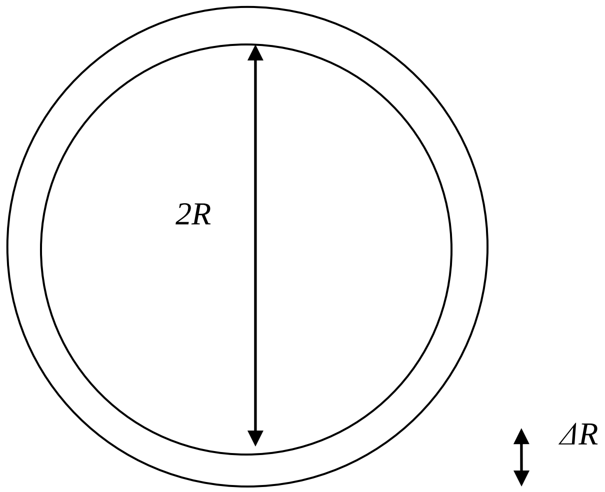
$$I \;=\; cMR^2 \tag{1}$$

where c is a numerical constant. This is obvious on dimensional grounds. Our task is to calculate the value of the constant c.

Eqn. (1) can be rewritten in terms of the areal density of the disc, ρ, measured in kilograms per square metre. The mass M is equal to $\pi R^2\rho$, so

$$I \;=\; c\pi\rho R^4 \tag{2}$$

Now consider two discs, equal in density, one of radius R and another whose radius is greater by a small increment ΔR. The two discs are illustrated concentrically below:



The moment of inertia of the inner disc, $I(R)$, is just $c\pi\rho R^4$, that being equation (2) above. For the moment of inertia of the outer disc, R must be replaced by $R + \Delta R$, and it may be written

$$I(R + \Delta R) \;=\; c\pi\rho(R + \Delta R)^4 \tag{3}$$

It is tedious but straightforward to multiply out $(R + \Delta R)^4$, and the result is

$$I(R + \Delta R) = c\pi\rho\,.(\,R^4 + 4R^3\Delta R + 6R^2(\Delta R)^2 + 4R(\Delta R)^3 + (\Delta R)^4\,) \tag{4}$$

If ΔR is very small compared with R, then each of the terms in the brackets on the right-hand side, following R^4, will be much smaller than the term preceding it. So, to a good approximation (to "first order" in ΔR, as we say), the later terms can be neglected and we can just write

$$I(R + \Delta R) = c\pi\rho\,.(\,R^4 + 4R^3\Delta R\,) \tag{5}$$

However, there is another way of obtaining the moment of the outer disc. We can simply take the moment of inertia of the inner disc, as in equation (2), and add on the moment of inertia of the ring (or "annulus") between the two discs. Since ΔR is small, the circumference of the ring is $2\pi R$. Multiplying the circumference by the thickness of the ring, ΔR, we obtain the ring's area $2\pi R.\Delta R$, and multiplying this by the density ρ gives the mass of the ring, $2\pi\rho R.\Delta R$. Since the ring is all at practically the same distance R from the centre (ΔR being very small) the ring's moment of inertia is obtained simply by multiplying its mass by the square of the distance R, giving $2\pi\rho R^3.\Delta R$.

So, we must have

$$I(R + \Delta R) = c\pi\rho R^4 + 2\pi\rho R^3.\Delta R \tag{6}$$

but for equations (5) and (6) both to be true, it must obviously be that

$$c\pi\rho.4R^3\Delta R = 2\pi\rho R^3\Delta R\,) \tag{7}$$

giving

$$c = ½ \tag{8}$$

which means, from equation (1), that

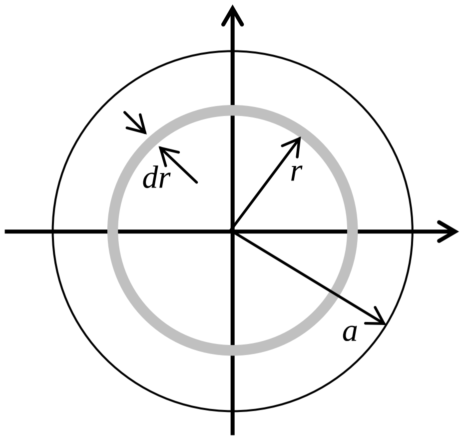
$$I = ½\,MR^2 \tag{9}$$

and this is the desired formula for the moment of inertia of a disc rotating around the axis through its centre and perpendicular to it.

A uniform circular disc: the method of integration

The previous result can be obtained straightforwardly using calculus, integrating over the volume of the disc.

We can take the element of volume to be a circular ring of radius r and width dr. Then



$$dV = 2\pi\, r\, t\, dr$$

where t is the thickness of the disk. The mass element is then

$$dm = \rho dV = 2\pi\rho\, r\, t\, dr$$

The moment of inertia is then

$$I = \int r^2 dm = \int r^2 \rho\, dV = \int_0^R r^2\, 2\pi\rho\, r\, t\, dr$$

so that

$$I \quad = \quad 2\pi\rho t \int_0^R r^3\, dr \quad = \quad 2\pi\rho t \left[\frac{r^4}{4}\right]_0^R \quad = \quad ½\, \pi\rho R^4 t$$

But the total mass of the disk is $M = \pi R^2 t\, \rho$. It follows, therefore, that the moment of inertia of the disk is

$$\boxed{I = \tfrac{1}{2} M R^2}$$

which is, of course, the same result as has already been found by a different method.

A thin spherical shell

What is the moment of inertia of a thin spherical shell rotating around an axis which passes through its centre?

Consider the shell as a collection of very small parts (labelled by *1, 2, 3, ..., i, i+1, i+2, ...*) with masses m_i and positions $\underline{r}_i$. In terms of orthogonal Cartesian coordinates x, y and z, the positions $\underline{r}_i$ may be represented by

$$\underline{r}_i \quad = \quad (\, x_i, y_i, z_i\,) \tag{1}$$

and the moment of inertia for rotation around the z-axis is

$$I_z = \sum_i m_i.(x_i^2 + y_i^2)$$

(2)

because, by Pythagoras' Theorem, $(x_i^2 + y_i^2)$ is equal to the square of the distance of the piece i from the z-axis. In fact, this equation is true for an object of any shape at all, not just a spherical shell. Similarly, the moments of inertia for rotation around the x-axis and the y-axis are

$$I_x = \sum_i m_i.(y_i^2 + z_i^2)$$

(3)

and

$$I_y = \sum_i m_i.(x_i^2 + z_i^2)$$

(4)

Adding all these together, we get

$$I_z + I_x + I_y = \sum_i m_i.\left(2x_i^2 + 2y_i^2 + 2z_i^2\right)$$

(5)

Because of the symmetry of a spherical shell, the moments of inertia around the x, y and z axes must be equal. The shell looks the same from all directions. So,

$$I_z = I_x = I_y$$

(6)

Also, in the particular case of a spherical shell, every part of the object is exactly the same distance (R, say) from the origin, so

$$R^2 = x_i^2 + y_i^2 + z_i^2$$

(7)

independently of i. This is the Theorem of Pythagoras in 3 dimensions. Putting equations (6) and (7) into (5), and using

$$\sum_i m_i = M$$

(8)

for the total mass of the shell, we get

$$3.I_z \quad = \quad 2MR^2 \tag{9}$$

so that

$$I_z \quad = \quad ⅔\ MR^2 \tag{10}$$

as our final result for the moment of inertia of a thin spherical shell of mass M and radius R.

Question 7a

What is the moment of inertia of a uniform solid sphere rotating around an axis which passes through its centre?

Solution: use basically the same technique that was used earlier to calculate the moment of inertia of a disc. If the sphere has a mass M and a radius R, then the moment of inertia must be proportional to MR^2. Express that fact in terms of the density of the sphere instead of its mass. Then, consider the moment of inertia of a sphere whose radius is greater than R by a small amount ΔR. The difference between the two must be the moment of inertia of a spherical shell of thickness Δ, which is given by the formula derived in the previous section. Deduce that the formula for the moment of inertia of a solid sphere is given by

$$I \quad = \quad ⅖\, MR^2$$

Some terminology: radius of gyration

The *radius of gyration, k,* is defined so that the moment of inertia of a body of mass *M*, rotating about its centre of mass is

$$I = Mk^2$$

This means that the radius of gyration may be calculated via

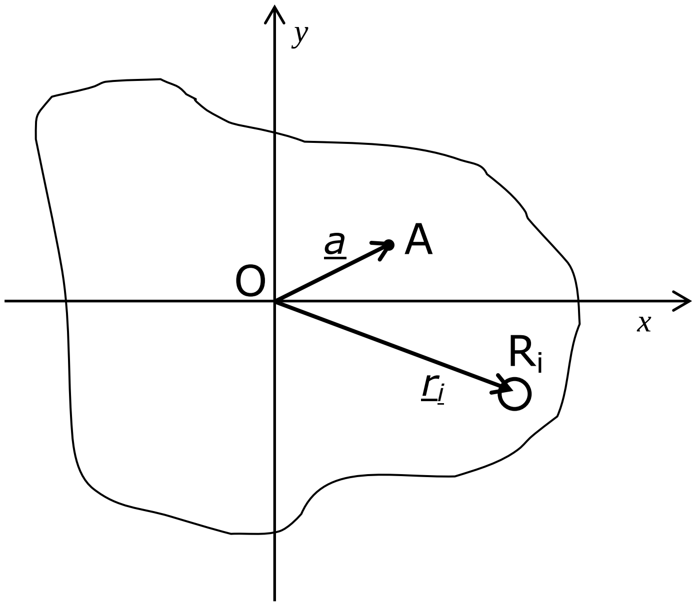
$$k^2 = \frac{1}{M}\int r^2 dm$$

Using results derived above and later below we find that the radii of gyration for some common shapes are:

$$k_{disc} = \frac{1}{\sqrt{2}} a \ ; \qquad k_{sphere} = \sqrt{\frac{2}{5}} a \ ; \qquad k_{rod} = \frac{1}{\sqrt{12}} L$$

The Parallel Axes Theorem

The "parallel axes" theorem is a very useful relationship between an object's the moments of inertia for rotation around two different axes, one of which passes through the object's centre of mass. The proof of the theorem is as follows:



A large irregularly-shaped object is illustrated above.

The origin of coordinates is the point O. Coordinate axes (x and y) are illustrated, but we shall not actually need to use them in the calculation below.

The origin O is chosen to be at the object's centre of mass.

We are going to calculate the moment of inertia for the rotation of the object around an axis which is perpendicular to the illustration and which passes through the point labelled A. The position vector of A (in other words, its displacement from the origin O) is $\underline{a}$.

The object is imagined to be broken up into a very large number of very small parts labelled R_1, R_2, R_3, R_4, R_5, and so on. In general, any of these parts is labelled as R_i, where the index i may take on any of the values 1, 2, 3, 4, 5, and so on. The position vector of R_i is denoted as $\underline{r}_i$. Let the mass of the part R_i be m_i.

It is important to understand that, because O is the centre of mass of all the parts R_i which make up this object, we have

$$\sum_i m_i \underline{r}_i = 0$$

This is so because, in any coordinate system, the centre of mass of a set of masses m_i at positions r_i is given by

$$\underline{r}_{CM} = \sum_i m_i \underline{r}_i \Big/ \sum_i m_i \tag{1}$$

or

$$\underline{r}_{CM} = \frac{1}{M} \sum_i m_i \underline{r}_i \tag{2}$$

where the total mass of the object is represented by M, so that

$$M = \sum_i m_i \tag{3}$$

However, because the origin O has been chosen to coincide with the centre of mass, the position vector $\underline{r}_{CM}$ of the centre of mass is zero. In other words, the centre of mass is at zero distance from the origin. Because $\underline{r}_{CM} = 0$, equation (1) above leads to

$$\sum_i m_i \underline{r}_i = 0 \tag{4}$$

To calculate the moment of inertia of the object when it is rotating around an axis through A, the mass m_i of each part R_i must be multiplied by the square of its distance from that axis, i.e. the distance (in the plane of the diagram) from A to R_i. The displacement vector from A to R_i is got by just subtracting their position vectors, $\underline{a}$ and $\underline{r}_i$, and so the square of the distance is

$$\left|\underline{r_i} - \underline{a}\right|^2$$

The moment of inertia around A is therefore

$$I_A = \sum_i m_i \left|\underline{r_i} - \underline{a}\right|^2$$

or

$$I_A = \sum_i m_i \left(r_i{}^2 - 2\underline{r_i} \cdot \underline{a} + a^2\right)$$

or

$$I_A = \sum_i m_i r_i{}^2 - \sum_i m_i . 2\underline{r_i} \cdot \underline{a} + \sum_i m_i a^2$$

and using equation (3) from the box above, this gives

$$I_A = \sum_i m_i r_i{}^2 - M.\underline{r_{CM}} \cdot \underline{a} + Ma^2$$

and because of equation (4) the middle term on the right hand side is zero, leaving

$$I_A = I_{CM} + Ma^2$$

This is the "parallel axes theorem". It means that, if we know the moment of inertia for the rotation of an object of mass M around an axis which passes through its centre of mass, then we can easily calculate its moment of inertia for rotation around any other parallel axis: we just add on Ma^2, where a is the distance between the two axes.

A uniform straight rod

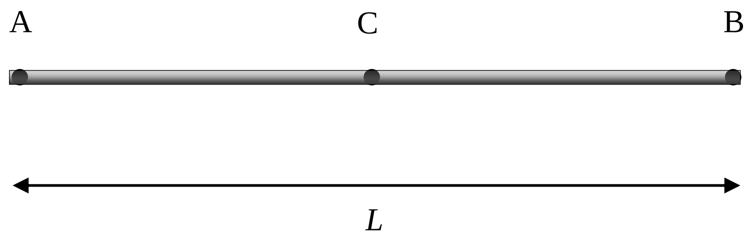
Although the moment of inertia of a thin uniform rod, rotating around an axis perpendicular to it, can be calculated easily by integration, another method of calculation is given here using the Parallel Axes Theorem. Suppose that $I(M, L)$ is the moment of inertia of a uniform rod of mass M and length L rotating around an axis which is perpendicular to the rod and passes through one of its endpoints.

The moment of inertia $I(M, L)$ must be proportional to M and to the square of L. That is to say,

$$I \quad = \quad cML^2 \tag{1}$$

where c is a numerical constant. This is obvious on dimensional grounds. What is the value of c?

The rod is illustrated below. It is understood that the length L is much greater than the thickness of the rod. For rotation around an axis passing through A or B, then, the moment of inertia is given by equation (1).



But what is the moment of inertia for rotation around an axis through the point C at the centre of the rod? The point C is the rod's centre of mass, so let us designate that moment of inertia by I_{CM}. Now, I_{CM} can be regarded as the sum of two parts: the moment of inertia of the right-hand half of the rod (CB, rotating around C) and that of the left-hand half of the rod (AC, also rotating around C). These are equal to one another, and each is the moment of inertia of a rod of mass *M/2* and length *L/2*. So, adding the two halves together, we get

$$I_{CM} \quad = \quad c.\left(\frac{M}{2}\right).\left(\frac{L}{2}\right)^2 \; + \; c.\left(\frac{M}{2}\right).\left(\frac{L}{2}\right)^2 \tag{2}$$

whence

$$I_{CM} \quad = \quad ¼\; cML^2 \tag{3}$$

The trick is now to use the parallel axes theorem to obtain, from equation (3), a formula for the moment of inertia of the whole rod around an axis through one of its endpoints (A or B). The parallel axes theorem can be written

$$I_A \quad = \quad I_{CM} \; + \; Ma^2 \tag{4}$$

In this instance, I_{CM} is given by equation (3) and the distance a between the two axes is simply the distance between the points A (or B) and C, that is to say *L/2*. Therefore,

$$I_A \quad = \quad ¼\, cML^2 \quad + \quad M\left(\frac{L}{2}\right)^2 \tag{5}$$

or

$$I_A \quad = \quad ¼\,(c+1)ML^2 \tag{6}$$

But, if our theory is consistent, this has to be equal to the expression in equation (1). Therefore,

$$cML^2 \quad = \quad ¼\,(c+1)ML^2 \tag{7}$$

from which it follows that

$$c \quad = \quad ⅓ \tag{8}$$

so finally we arrive at

$$I \quad = \quad ⅓\,\mathrm{M}L^2 \tag{9}$$

as the formula for the moment of inertia of a thin uniform rod rotating around an axis perpendicular to it and passing through one end.

Question 7b

What is the moment of inertia of a thin uniform rod, of length L, rotating around its centre of mass?

Solution: the required moment of inertia is what was called I_{CM} *in the section above. Its value is given by equation (3) above, with the value of* c *being ⅓ (eqn. (8) above)/ So, for a rod rotating around its centre,*

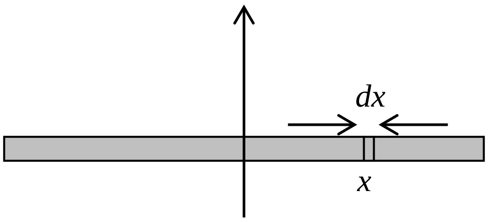
$$I \quad = \quad \frac{1}{12}\,\mathrm{M}L^2$$

Question 7c

If you are familiar with integral calculus, verify the solution of question 7b by integrating over the length of the rod.

Solution: let ρ *be the mass per unit length of the rod. We take the mass element to be that portion of the rod between* x *and* $x+dx$*, then*

$$dm = \rho\, dx$$



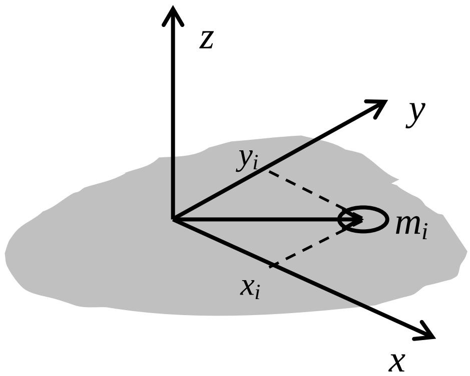
The moment of inertia is then

$$I \;=\; \int_{-L/2}^{L/2} x^2 dx \;=\; \int_{-L/2}^{L/2} x^2 \rho dx \;=\; ⅓\,\rho[x^3]_{-L/2}^{L/2} \;=\; ⅓\,\rho\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] \;=\; \frac{1}{12}\,\rho L^2$$

The total mass is $M = \rho L$*, so that* $I = \frac{1}{12} ML^2$*.*

The perpendicular axes theorem

The perpendicular axes theorem is a useful rule applicable to objects which are thin and flat. Such an object is sometimes called a "lamina".



Consider a lamina rotating around an axis which is perpendicular to itself. We will use a coordinate system such that the axis of rotation is the *z* axis and the lamina is in the *x-y* plane as shown in the diagram.

The fragment *i* has a mass m_i and coordinates x_i and y_i. ($z_i = 0$ because the lamina is a flat object in the *x-y* plane.)

The moment of inertia of the body is

$$I_z \;=\; \sum_i m_i\,.(x_i{}^2 \;+\; y_i{}^2)$$

because $(x_i{}^2 \;+\; y_i{}^2)$ is (by Pythagoras' Theorem) the square of the distance of the fragment *i* from the axis of rotation, i.e. from the *z* axis. But this can be split into two sums:

$$I_z \;=\; \sum_i m_i\,x_i{}^2 \;+\; \sum_i m_i\,y_i{}^2$$

But the first sum is just the same as the moment of inertia for rotation around the y axis, and the second sum is the moment of inertia for rotation around the x axis. So, switching the order of the two sums, we can write

$$I_z = I_x + I_y$$

and this is the perpendicular axes theorem.

Question 7d

A shop sign consists of a circular disc of mass *m* and radius *a* hinged along a tangent as shown in the sketch. Find a formula for the moment of inertia of the sign when it rotates around the hinge.

Solution: it was shown earlier that a disc rotating about an axis passing through its centre, perpendicular to its plane, has a moment of inertia which, in terms of m and a, is

$$I_z = \tfrac{1}{2} m a^2$$

Using the perpendicular axes theorem we have

$$I_z = I_x + I_y$$

and by symmetry

$$I_x = I_y$$

It follows that

$$I_x = I_y = \tfrac{1}{2} I_z = \tfrac{1}{4} m a^2$$

Here I_x and I_y would be the moments of inertia for rotation around axes which are in the plane of the disc and pass through its centre. To find the moment of inertia for the disc rotating about the tangential axis illustrated, which is offset from the disc's centre by a distance equal to the radius a*, it is necessary to use the parallel axes theorem:*

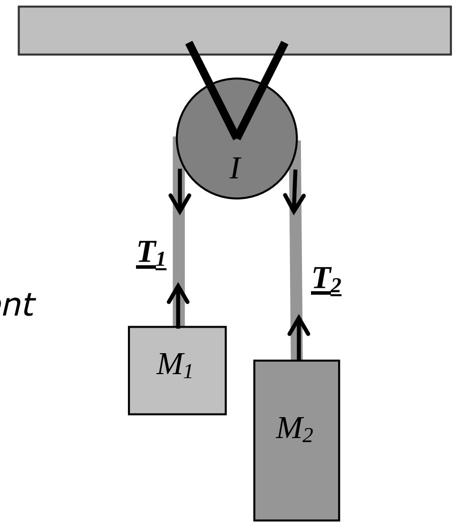
$$I_{\tan} = I_x + m a^2 = \tfrac{1}{4} m a^2 + m a^2 = \tfrac{5}{4} m a^2$$

Moments of inertia for objects of various shapes

For a useful summary of the formulae for the moments of inertia of objects of a number of different shapes, see Wikipedia.

Question 7e

Two masses M_1 and M_2 are connected by a light inextensible cord that passes over a cylindrical pulley of mass m and radius r. The cord has no tendency to slip on the pulley, so tensions in the two portions of the cord are different, T_1 and T_2, as show in the diagram. Find the linear acceleration of the masses once they are released.



Solution: firstly we write down the moment of inertia of the pulley:

$$I = \tfrac{1}{2} m r^2$$

Now the torque on the pulley is

$$\Gamma = (T_2 - T_1) r$$

and α, the angular acceleration of the pulley, is given by the rotation counterpart of Newton's second law:

$$\Gamma = I\alpha$$

Substituting for the torque and the moment of inertia gives

$$\Gamma = (T_2 - T_1) r = I\alpha = ½\, m r^2 \alpha$$

Now we can relate the angular acceleration α to the linear acceleration a using $a = r\alpha$. If we insert $\alpha = a/r$ into the torque equation and divide both sides by the common factor of r, we find:

$$T_2 - T_1 = \tfrac{1}{2} m a \qquad (i)$$

There are also the equations of motion ("F = ma") for the two masses:

$$M_2 g - T_2 = M_2 a \qquad (ii)$$

and

$$T_1 - M_1 g = M_1 a \qquad (iii)$$

If we add equations (ii) and (iii) we get

$$M_2 g - M_1 g - (T_2 - T_1) = (M_1 + M_2) a \, .$$

We now substitute for $T_2 - T_1$ from (i) and find:

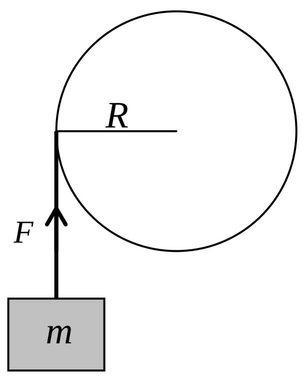
$$(M_2 - M_1) g = (M_1 + M_2 + \tfrac{1}{2} m) a \, .$$

Hence the acceleration of the masses is

$$a = \frac{(M_2 - M_1) g}{M_1 + M_2 + m/2}$$

7.3 Work and Power in Rotational Motion

In the sketch a block of mass *m* is being raised at a constant rate by a cable that winds around a rotating drum with radius *R*. Since the block is not being accelerated, the tension in the cable *F* is equal to the weight of the block *mg*. Suppose the block is raised through a distance *s*. The work done by the force *F* is then just the force times the displacement:



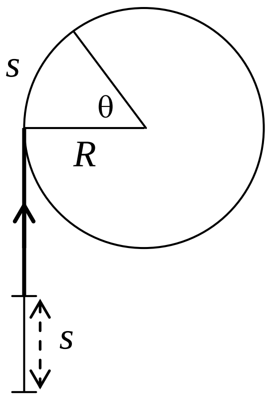
$$W = Fs$$

Now we can relate the distance *s* to the angular displacement θ by

$$s = R\theta$$

(see diagram following) and inserting this into the work equation gives

$$W = Fs = FR\theta = \Gamma\theta$$



The work dome in rotational motion is the product of the torque and the angular displacement.

Power is the rate at which work is done, and we have already shown that

$$P = \frac{dW}{d\,t} = F\,\frac{ds}{dt} = Fv$$

where v is the velocity with which the block is being raised. The angular velocity ω is related to the linear velocity v by $v = R\omega$. It follows therefore that for rotational motion

$$P = \frac{dW}{d\,t} = Fv = FR\omega = \Gamma\,\omega$$

Power exerted in rotational motion is the product of the torque and the angular velocity.

7.4 Rotational kinetic energy

Suppose a body with moment of inertia I is rotating at an angular velocity ω_i, and is then acted on by a constant torque Γ. In this case the angular velocity should increase linearly with time, the angular acceleration being given by $\Gamma = I\alpha$. From the equations for rotational motion we can work out the angular velocity following a certain angular displacement θ:

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta$$

It follows that the work done by the torque in acting over this displacement is

$$W = \Gamma\theta = I\alpha\theta = \tfrac{1}{2} I(\omega_f^2 - \omega_i^2)$$

We recognise that the work done has been stored as a form of kinetic energy of the body. We deduce that

$$\boxed{\text{Rotational kinetic energy} = \tfrac{1}{2} I \omega^2}$$

The definition of kinetic energy for a rotating body, $\tfrac{1}{2} I\omega^2$, has the same form as for linear motion, $\tfrac{1}{2} mv^2$, with the moment of inertia playing the part of mass and the angular velocity ω substituting for the linear velocity v.

The kinetic energy of a rolling wheel

As a wheel rolls along a surface there are two components to the kinetic energy, one due to the linear velocity of the centre of mass,

$$KE_{lin} = \tfrac{1}{2} mv^2$$

and the other due to the rotational kinetic energy,

$$KE_{rot} = \tfrac{1}{2} I\omega^2$$

The condition for rolling without slippage is that $v = \omega r$, so the total kinetic energy is

$$KE_{TOT} = \tfrac{1}{2} mv^2 + \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\left(m + \frac{I}{r^2}\right)v^2$$

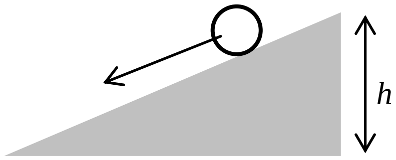
In the case of a solid disk, where $I = \tfrac{1}{2} mr^2$,

$$KE_{tot} = \tfrac{3}{4} mv^2$$

while for a hoop-like wheel, where all the mass is on the rim, $I = mr^2$ and

$$KE_{tot} = mv^2$$

Question 7f



Three objects: a solid sphere, a solid cylinder and a hoop are rolled down an incline. Each has the same mass *M* and radius *R* and they all travel through the same vertical distance *h*. What are their linear velocities, and how do these compare to the velocity of a cubic block that slides down the same incline, without friction?

Solution: we use the energy method here: the decrease in the PE of each object is Mgh*. This must equal the increase in the total KE*

$$\tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\left(M + \frac{I}{R^2}\right) v^2$$

It follows that

$$\tfrac{1}{2}\left(M + \frac{I}{R^2}\right) v^2 = Mgh$$

so that

$$v^2 = \frac{2Mgh}{M + I/R^2}$$

For the sphere, $I = \frac{2}{5}MR^2$*, so* $v_{sphere} = \sqrt{10/7gh} = 1.195\sqrt{gh}$

For the cylinder $I = \frac{1}{2}MR^2$*, so* $v_{cylinder} = \sqrt{4/3gh} = 1.155\sqrt{gh}$

For the hoop $I = MR^2$*, and so* $v_{hoop} = \sqrt{gh}$

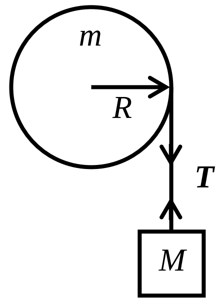
For the block that slides without friction: $v^2 = 2gh$*, so*

$$v_{block} = \sqrt{2gh} = 1.414\sqrt{gh}$$

The block travels fastest, followed by the sphere!

Question 7g

A block of mass M is suspended from a cable which wraps around a cylindrical drum of mass m and radius R. The block falls from rest. What is the velocity of the block after it has fallen, from rest, through a height h?



Solution: method 1

The equation for the linear acceleration of the mass M is

$$Mg - T = Ma \tag{i}$$

The equation for the angular acceleration of the drum is

$$\Gamma = TR = I\alpha = \tfrac{1}{2}mR^2\alpha \tag{ii}$$

From (ii), since $\alpha = a/R$, *it follows that*

$$T = \frac{1}{R}\tfrac{1}{2}mR^2\frac{a}{R} = \tfrac{1}{2}ma$$

We now substitute for T in eq.(i):

$$Mg - \tfrac{1}{2}ma = Ma$$

leading to

$$Mg = (M + \tfrac{1}{2}m)a$$

and so the linear acceleration is:

$$a = \frac{M}{M + \tfrac{1}{2}m}g$$

We now use the equation: $v^2 = u^2 + 2as$, *with u = 0 and s = h:*

$$v^2 = 2ah = \frac{M}{M + \tfrac{1}{2}m}2gh$$

Dividing top and bottom by M and taking the square root gives:

$$v = \sqrt{\frac{2gh}{1 + m/2M}}$$

Solution: method 2

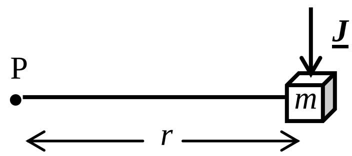
The second approach uses conservation of energy for the system: the increase in the kinetic energy (linear + rotational) must be equal to the decrease in the potential energy of the mass M when it falls through a distance h. It follows then that:

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = Mgh.$$

Now we only need to substitute $I = ½mR^2$ *and use the relation* $\omega = v/R$ *and the same answer results! This is clearly a much neater route to the solution.*

7.5 Angular momentum and angular impulse

Consider a body with mass *m* attached to a light rod, of length *r*, pivoted at point P. Suppose the body, initially at rest, experiences an impulse *J*, resulting from a force *F* applied for a time Δt, so that $J = F\Delta t$.



From Newton's laws, the change in the momentum of the body is equal to the impulse: $\Delta(mv) = J$.

However, since the mass is pivoted, it begins to rotate at an angular velocity ω given by $v = \omega r$. We can think of this rotational motion as resulting from an *angular impulse*, which is the *moment* of the linear impulse *J*:

$$Jr = (F \times r)\Delta t = \Gamma \Delta t.$$

An angular impulse is the product of a torque Γ *and the time* Δt *for which it acts.*

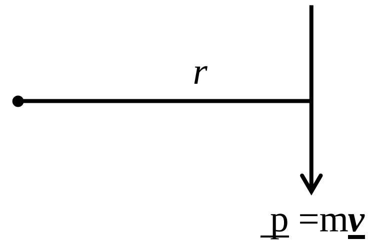
The result of the angular impulse is to produce a rotation about the pivot:

$$Jr = \Gamma \Delta t = \Delta(mvr) = \Delta(m\omega r^2)$$

This equation says that the effect of an angular impulse is to produce a change in the quantity mvr = mωr^2. This quantity is called the *angular momentum* of the body.

The angular impulse is equal to the change in the angular momentum of the system.

Angular momentum is the product of the *momentum* of a moving object and the perpendicular distance between the line of motion and the axis of rotation.



The usual symbol for angular momentum is *L* and it is measured in units of *kg m^2 s^{-1}* or *N m s*.

The angular momentum of a body of mass m rotating around an axis at a distance r with angular velocity ω *is given by* $L = m\omega r^2$

This definition works for a point mass, but what happens for an extended object? Suppose a body is subjected to a torque Γ for a time *t*. If the body has an initial angular velocity ω_i, it will be subjected to an angular acceleration α, so that after a time *t* its angular velocity will have increased to ω_f, where

$$\omega_f = \omega_i + \alpha t$$

From Newton's 2nd law for rotational motion: $\Gamma = I\alpha$, where I is the moment of inertia of the body. It follows that

$$\Gamma = I\alpha = I\frac{(\omega_f - \omega_i)}{t}$$

and so the angular impulse is

$$\mathit{\Gamma} t \;=\; I\left(\omega_f - \omega_i\right) \;=\; \Delta L$$

This gives another representation of the angular momentum:

Angular momentum is the product of the moment of inertia and the angular velocity: $L = I\omega$

Clearly this is the rotational equivalent of linear momentum *mv,* since ω is the angular velocity and I plays a role in rotational motion equivalent to mass in linear motion.

The rate of change of angular momentum

If we suppose that the torque is applied for a short time, Δt, it would follow from above that $\Gamma\Delta t = \Delta L$, so in the limit of $\Delta t \to 0$ we have that the rate of change of the angular momentum is equal to the net applied torque acting on the system:

$$\boxed{\Gamma = \frac{dL}{dt}}$$

The conservation of angular momentum

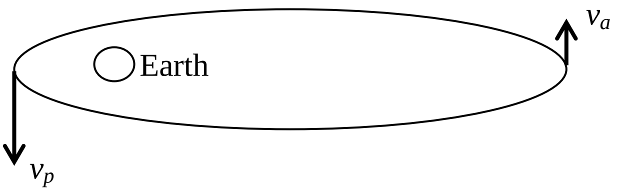
If the system is not subjected to external torques, then, from above, the rate of change of the angular momentum must be zero. It follows that the angular momentum is then a constant, independent of time. This is a basic principle of physics, as fundamental as the conservation of linear momentum:

If there are no external torques acting on the system, then the angular momentum is conserved.

A particular example where conservation of angular momentum applies is for a system acted on by *central forces*, for example, consider the planets in orbit around the Sun. A planet is kept in its orbits by the gravitational attraction between the Sun and the planet. This acts along the line joining the two bodies, so there is no torque involved. The angular momentum is therefore conserved.

Question 7h

A satellite is on a highly elliptical orbit around the Earth, where the distance of closest approach (perigee) is 10000 km and the furthest distance (apogee) is 15000 km, both measured from the centre of the Earth.
What is the ratio of the orbital speeds at perigee and apogee?



Solution: since angular momentum is conserved,

$$L = mv_p r_p = mv_a r_a$$

where subscript p means perigee and subscript a means apogee. The satellite will be travelling faster at perigee than at apogee:

$$\frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{15000}{10000} = 1.5$$

Question 7i

The Sun has a radius of 695000 km and a mass of 2×10^{30} kg. It rotates on its axis once every 25 days. Suppose the Sun collapsed into a neutron star with the same mass but a radius of only 10 km. How fast would it rotate?

Solution: assuming the Sun to have a uniform density, we can take its moment of inertia to be

$$I = \tfrac{2}{5}MR^2$$

Its angular momentum is then

$$L = I\omega$$

Since angular momentum is conserved in the collapse,

$$I_i\omega_i = I_f\omega_f$$

It follows that the ratio of the angular velocity after collapse to that before the collapse would be:

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f} = \frac{R_i^2}{R_f^2} = \frac{(695000)^2}{10^2} = 4.83 \times 10^9$$

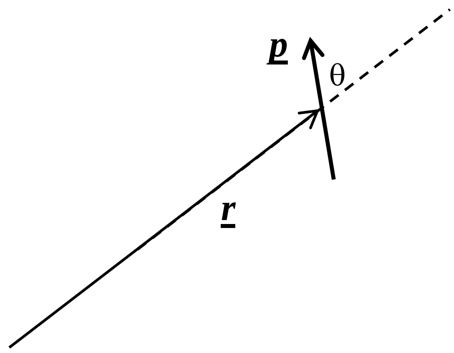
so that

$$\omega_f = 4.83 \times 10^9 \times \omega_i = 4.83 \times 10^9 \times \frac{2\pi}{25 \times 24 \times 3600} rad.s^{-1} = 14051 rad.s^{-1}$$

This corresponds to 2236 revolutions per second, or a period of 0.45 milliseconds.

Angular momentum and torque as vector quantities

Suppose a body is at a position $\underline{r}$ and has a linear momentum $\underline{p} = m\underline{v}$. From our definition of the angular momentum, it is the component of the momentum at right angles to the radius vector $\underline{r}$ that produces the "turning moment".



The angular momentum is therefore

$$L = mvr\sin\theta$$

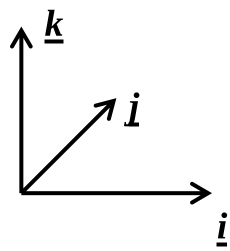
,

where θ is the angle between $\underline{r}$ and $\underline{p}$, as shown in the diagram. Another way of writing this is in terms of a vector product (or cross product):

$$\underline{L} = \underline{r} \times \underline{p}$$

The *order* in a vector product is important since

$$\underline{A} \times \underline{B} = -\underline{B} \times \underline{A}$$

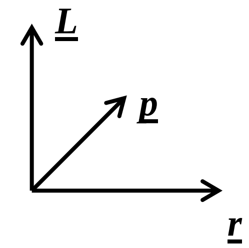


To remember the convention,use a right-anded set of axes, with unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ along the *x*, *y*, and *z* axes. Then,

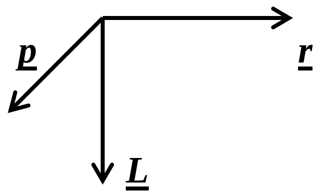
$$\underline{i} = \underline{j} \times \underline{k}$$

This gives a "right hand rule": imagine your right index finger pointing along *z*, and your right thumb aligned along *x*. Then a clockwise rotation of the thumb from *x* to *y* is like the motion of a screwdriver, driving a screw along the positive *z* axis. Conversely, going clockwise from *y* to *x* would require the index finger to point along the *negative z* direction. This corresponds to $\underline{j} \times \underline{i} = -\underline{k}$.

Using this rule, since $\underline{L} = \underline{r} \times \underline{p}$, we can see from the sketch on the right how to draw the direction of the angular momentum $\underline{L}$ vector when the position vector $\underline{r}$ and the momentum vector $\underline{p}$ are perpendicular to each other.



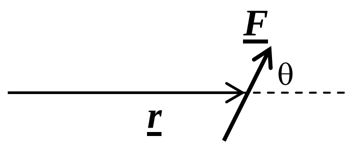
Reversing the direction of $\underline{p}$ would reverse $\underline{L}$ so that it pointed *downwards*, as shown in the second sketch.



The conservation of angular momentum implies that both the magnitude and direction of $\underline{L}$ are constant with time. This means, for example, that the planets have to move in fixed plane that passes through the Sun.

In exactly the same way we can see that a torque may also be represented by a vector product: the magnitude of the torque depends on the product of the radial distance from the pivot and the component of the force normal to the radius vector:

$$\Gamma = r \times F \sin\theta .$$



Clearly we can then write a torque as:

$$\underline{\Gamma} = \underline{r} \times \underline{p}$$

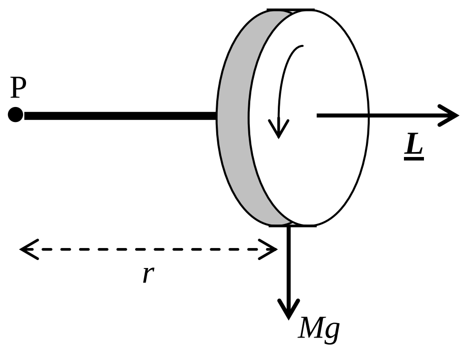
The direction of the torque vector is also given by the right-hand rule, so that in the sketch above the torque points out of the image. It makes sense that a clockwise torque has an opposite sign to an anticlockwise torque, since if two equal and opposite torques are applied to a system, they would cancel each other out.

It follows that when we write down the equation that relates the rate of change of angular momentum to the applied torque, we should really write this as a *vector* equation:

$$\underline{\Gamma} = \frac{d\underline{L}}{dt}$$

The new information that this equation gives us is that direction in which the angular momentum changes is parallel to the direction of the applied torque.

The precession of a gyroscope



A gyroscope is a flywheel supported on a pivot so that we may investigate how it is affected by external torques. In the sketch the flywheel is rotating clockwise when viewed from the pivot P, so the angular momentum vector $\underline{L}$ points along the axis, away from the pivot, as shown. Since the axis is horizontal, and the gyroscope is only supported at the pivot, there will be a torque $\Gamma = Mgr$ due to the weight of the gyroscope. This is a clockwise torque, so its direction is into the plane of the paper, normal to the angular momentum $\underline{L}$.

From what we saw in the previous subsection, the effect of a torque is to produce a change in the angular momentum in a direction parallel to the direction of the torque:

$$\Delta\underline{L} = \underline{\Gamma}\Delta t$$

This moves the direction of $\underline{L}$ in a perpendicular direction, into the paper. Since the torque always remains perpendicular to $\underline{L}$, the angular momentum continues to follow the direction of the torque, and the whole gyroscope precesses on its pivot. The axis of the gyroscope rotates around in a circle (anticlockwise, viewed from the top). If we spun the flywheel in an anticlockwise direction, the direction of the precession would be reversed.

7.6 Comparing linear motion and rotation

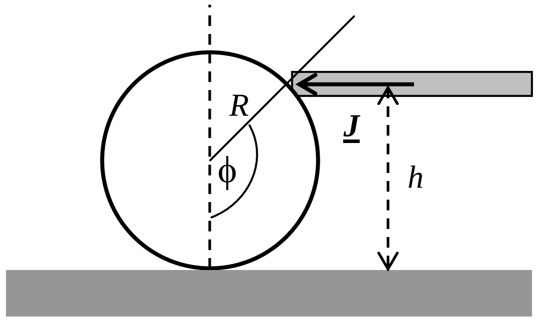
The table below summarises analogies between linear motion and rotation:

Linear motion		***Rotation***	
Mass	m	I	Moment of Inertia
Velocity	v	ω	Angular velocity
Acceleration	a	α	Angular acceleration
Force	F	Γ	Torque
Newton's 2nd law	$F = ma$	$\Gamma = I\alpha$	
Work	Fs	$\Gamma\theta$	Work
Power	Fv	$\Gamma\omega$	Power
Kinetic Energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$	Kinetic Energy
Impulse	$F\,t$	Γt	Angular
Momentum	$p = mv$	$L = I\omega$	Angular
Force = rate of change of momentum	$F = \frac{dp}{d\,t}$	$\Gamma = \frac{dL}{d\,t}$	Torque = rate of change of angular momentum

7.7 Examples involving both linear and angular impulses

Question 7j

A snooker ball has a radius R. It is set into motion by a sharp horizontal impulse from the cue. At what height above the table should the player strike the ball so that in the subsequent motion the ball rolls without slipping?



Solution: suppose the cue strikes the ball at a height h, imparting an impulse J. The change in the linear momentum is equal to the linear impulse:

$$J = mv \qquad \text{(i)}$$

However, the impulse also produces a rotation of the ball about the centre of mass.

The angular impulse is

$$J(h-R)$$

but the height $h = R(1-\cos\phi)$*, so we can write the angular impulse as*

$$-JR\cos\phi$$

(ii)

The change in the angular momentum must be equal to the angular impulse, so that

$$I\omega = -JR\cos\phi$$

(iii)

For a solid sphere the moment of inertia is $I = \frac{2}{5}mR^2$*. From (iii), therefore, we have*

$$\omega = -\frac{5J\cos\phi}{2mR}$$

(iv)

Now, the condition for rolling is that $v = \omega R$*. We can then compare v from (i) to* ωR *from (iv):*

$$v = \frac{J}{m} = \omega R = -\frac{5J\cos\phi}{2m}.$$

From this equation we find

$$\cos\phi = -\tfrac{2}{5}$$

This is the condition for the ball to roll without slipping: it is only for this value of ϕ *that the linear and angular impulse are in the ratio required to impart linear and angular velocities in the correct ratio for rolling. Since*

$$h = R(1-\cos\phi)$$

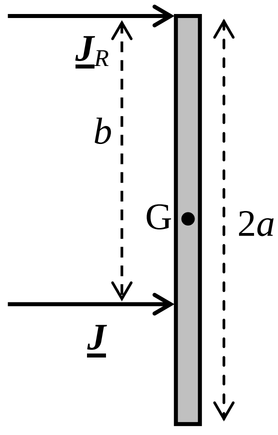
the required height of the cue is

$$h = R(1-\cos\phi) = R(1-(-2/5)) = 7R/5 = 1.4R.$$

Question 7k

Show that the "sweet spot" on a cricket bat is 2/3 of the way down the blade from the handle. If the bat strikes a ball at this point the batsman feels no recoil from the stroke, whereas if the ball is struck above or below this point, the bat jars the batsman's wrists.

Solution: the diagram shows the bat sideways on. The blade is of length 2a and the impact of the ball is represented as the linear impulse $\underline{J}$ *at a distance b below the handle. The recoil of the bat felt by the batsman is represented by the impulse* $\underline{J}_R$ *at the handle.*



The total impulse is equal to the change in momentum of the bat so that

$$J + J_R = Mv \qquad (i)$$

Here M is the mass of the bat and v is the linear velocity of the centre of mass after the impact.

The angular impulse gives rise to a change in the angular momentum, with an angular velocity ω *around the handle:*

$$bJ = I_H \omega \tag{ii}$$

where I_H *is the moment of inertia of the bat about the handle end.*

Assuming that the bat can be represented approximately as a rectangular plate of length 2a, the moment of inertia about the centre of mass G is

$$I_G = \tfrac{1}{3} M a^2$$

so the moment of inertia about the handle is, using the parallel axes theorem,

$$I_H = I_G + M a^2 = \tfrac{4}{3} M a^2$$

The condition for rotation of the centre of mass about the handle connects the linear velocity of G and the angular velocity: $v = a\omega$*.*
It follows from (ii) that

$$bJ = I_H \omega = \tfrac{4}{3} M a^2 \frac{v}{a} = \tfrac{4}{3} M a v \tag{iii}$$

Substituting for Mv *from (i) we find*

$$bJ \;=\; \frac{4}{3}\, a(J + J_R)$$

so the recoil impulse is:

$$J_R = J\left[\frac{3b}{4a} - 1\right] \tag{iv}$$

The recoil vanishes when

$$3b = 4a$$

i.e. when

$$b = \tfrac{4}{3} a$$

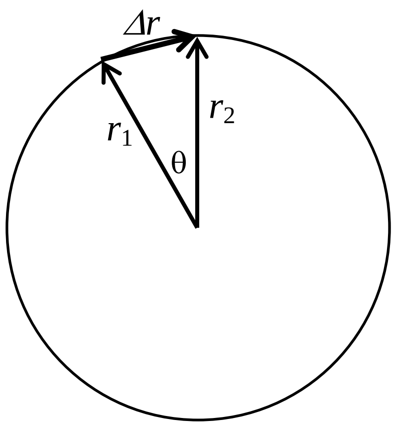
This corresponds to 2/3 of the way down the blade from the handle.

If the ball strikes the bat above this point, it can be seen from (iv) that J_R *is negative, i.e. in a direction opposite to the impulse from the ball. When the ball strikes below the sweet spot, the recoil at the handle is positive, in the same direction as the impulse from the ball.*

7.8 Motion in a circle

Here, we scrutinise the motion of a body constrained to move on a circular path at a constant angular velocity. The key feature to understand here is that while the magnitude of the linear velocity of the body is constant, the direction of the velocity keeps changing, so the body is subject to an acceleration.

Consider the positions $\underline{r}_1$ and $\underline{r}_2$ of a body in circular motion at two successive instants. The movement of the body between these instants is represented by the vector $\Delta\underline{r} = \underline{r}_2 - \underline{r}_1$, as illustrated:

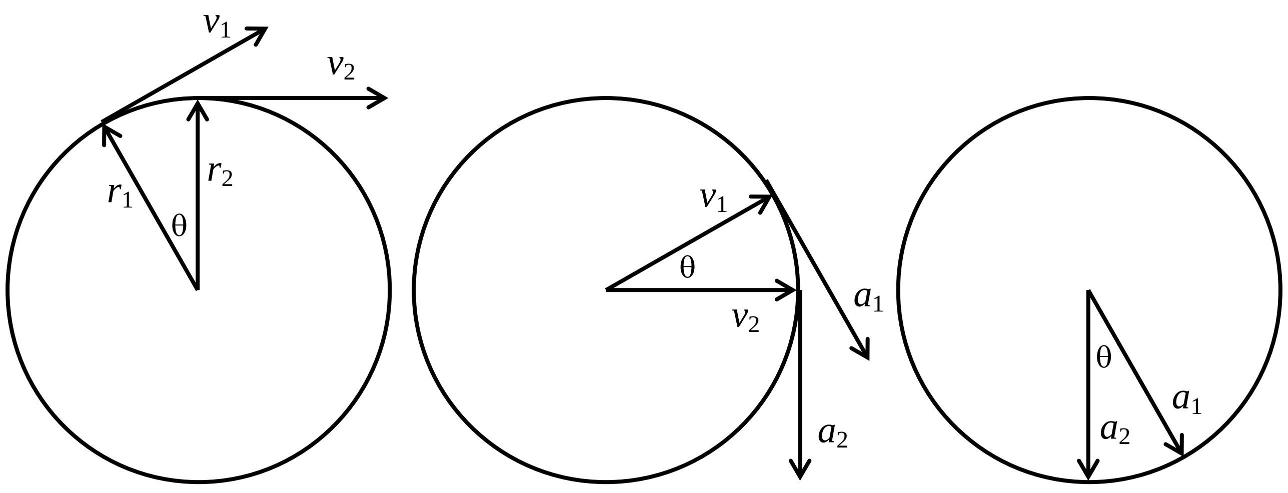


If the angle θ is small (i.e. the time interval is small) we may approximate the chord Δr by the arc $r\theta$, where r is the radius of the circle. Now, since the angular velocity is constant, we can write $\theta = \omega t$. It follows that the linear velocity of the body is

$$v = \frac{\Delta r}{t} = \frac{r\theta}{t} = \frac{r\omega t}{t} = \omega r$$

In the limit of $\theta \to 0$ it is clear that the instantaneous velocity of the body is perpendicular to the radius vector.

Now we need to look at the way the linear velocity changes with time. In the figure following, the position vectors and the instantaneous velocities $\underline{v}_1$ and $\underline{v}_2$ are shown in on the left.



Motion in a circle showing (left to right) the position, velocity and acceleration vectors.

In the central diagram, the velocity vectors are drawn with a common origin: these vectors also rotate on a circle at the same rate, but they are rotated by $\pi/2$ in phase by comparison with the position vectors, since each velocity vector is normal to its corresponding position vector. The change in velocity is $\Delta\underline{v} = \underline{v}_2 - \underline{v}_1$, and using the same approach as above, if θ is small we may approximate the chord Δv by the arc $v\theta$, where v, the magnitude of the velocity, is the radius of the circle in the central part of the diagram. Now we can write $\theta = \omega t$, and it follows that the linear acceleration of the body is

$$a = \frac{\Delta v}{t} = \frac{v\theta}{t} = \frac{v\omega t}{t} = \omega v \, .$$

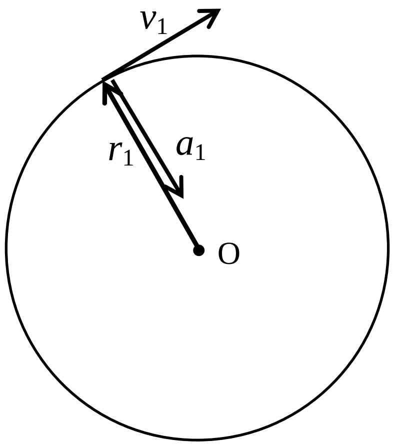
Using the result $v = \omega r$, it follows that the acceleration of the body is

$$\boxed{a = \omega v = \omega^2 r = \frac{v^2}{r}}$$

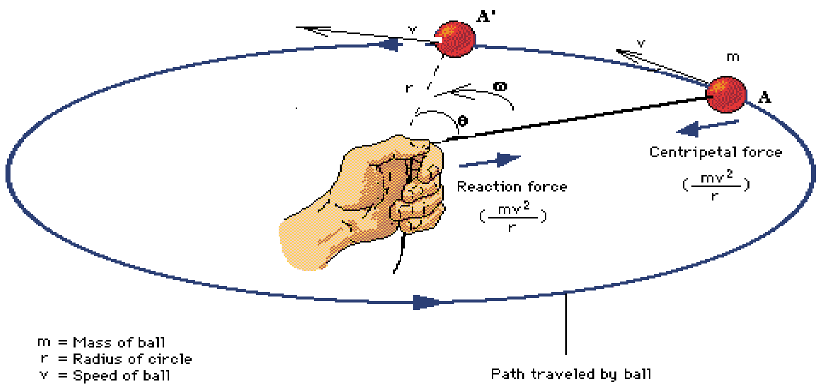
Again it is clear that in the limit of $\theta \to 0$ the instantaneous acceleration of the body is normal to the velocity vector. On the right of the diagram above, the acceleration vectors have been drawn with common origins. It may be that the acceleration vector is always *anti-parallel to the position vector*.

If we now plot the instantaneous position, velocity and acceleration vectors on the same diagram, as in the diagram below, we see that the acceleration vector points along the radius vector towards the origin O.

For this reason the acceleration in circular motion is called *centripetal acceleration* (centripetal ≡ seeking the centre).



For an object to move in a circle, a force must be applied to provide the centripetal acceleration. For example a mass on the end of a string can be constrained to move in a circle by the tension in the string:



In the case of a racing car travelling on a circular race track, the centripetal acceleration is provided by the sideways frictional force of the tyres on the track. For a satellite orbiting the Earth, or a planet orbiting the Sun, it is the gravitational attraction that provides the centripetal acceleration.

Question 7l

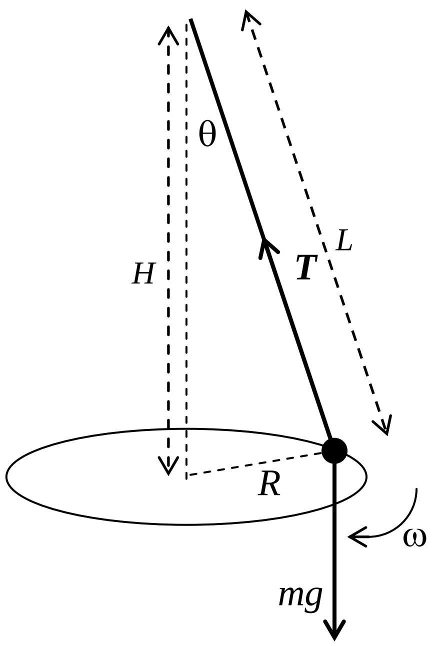
A mass of 2 kg is attached to a cord 25 cm long and whirled in a horizontal circle at 200 rev min^{-1}. What is the tension in the cord?

Solution: the centripetal force is

$$F = m\omega^2 r = 2\times\left(\frac{200\times 2\pi}{60}\right)^2\times 0.25 = 219N .$$

The conical pendulum

The illustration below shows a small bob of mass *m* attached to a cord of length *L.* It rotates in a horizontal plane at a constant angular velocity.



This is called a conical pendulum, since the cord moves on the surface of a cone. In this case the centripetal acceleration is provided by the horizontal component of the tension in the cord: the equation of motion of the mass is

$$T\sin\theta = \frac{mv^2}{R} = m\omega^2 R \tag{i}$$

Resolving the forces vertically gives

$$T\cos\theta = mg$$

We also have

$$\frac{R}{L} = \sin\theta$$

so that from (i)

$$T\frac{R}{L} = m\omega^2 R$$

and therefore

$$T = m\omega^2 L = \frac{mg}{\cos\theta}$$

The angular velocity of the pendulum is therefore

$$\omega = \sqrt{\frac{g}{L\cos\theta}} = \sqrt{\frac{g}{H}}$$

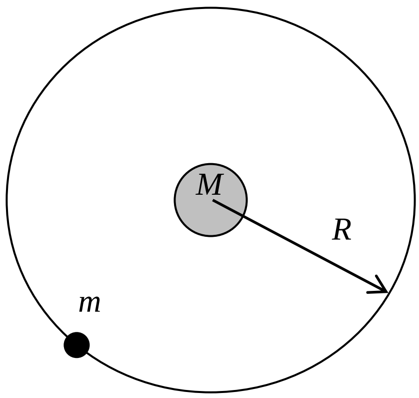
where *H* is the vertical height of the mass below the pivot point. The period of the conical pendulum is then:

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{H}{g}}$$

Note that the period is independent of the mass of the body.

The orbits of planets, moons and satellites

Suppose a planet of mass *M* has a moon of mass *m* in a circular orbit of radius *R*. Here the centripetal force that keeps the moon in its orbit is provided by the gravitational attraction between the two bodies.



From Newton's law of universal gravitation the force is

$$F = G\frac{Mm}{R^2}$$

where G is the gravitational constant (6.67×10^{-11} N m^2 kg^{-2}) The equation of motion of the moon is

$$m\omega^2 R = G\frac{Mm}{R^2}$$

It follows that the angular velocity is

$$\omega = \sqrt{\frac{GM}{R^3}}$$

and therefore that the period of the orbit is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}}$$

This proves one of Kepler's law's of planetary motion, that the square of the period of an orbit is proportional to the cube of the radius.

Question 7m

Show that Newton's law of gravitation can consistently explain both the acceleration of a falling apple and the period of the Moon's orbit around the Earth. [The radius of the Earth is 6371 km and the radius of the Moon's orbit is 3.82×10^5 km.].

Solution: firstly the force on an apple of mass m at the Earth's surface is

$$F = G\frac{Mm}{{R_E}^2}$$

where M is the mass of the Earth and R_E is the radius of the Earth. The acceleration of the apple is then

$$g = F/m = GM/R_E^2$$

Now g = 9.8 m s^{-2}, and we are told that R_E =6.371 $\times$ 10^6 m. It follows that

$$GM = gR_E^2 = 9.8 \times (6.371 \times 10^6)^2 = 3.977 \times 10^{14} m^3 s^{-2}.$$

From the value of G the gravitational constant ($6.67 \times 10^{-11} Nm^2 kg^{-2}$) we may deduce the mass of the Earth:

$$M = GM / G = 3.977 \times 10^{14} / 6.67 \times 10^{-11} = 5.96 \times 10^{24} kg.$$

Now we can check this value for GM from the radius of the Moon's orbit, which has a period of 27.3 days. From above the period of the orbit is

$$\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R^3}{GM}}.$$

We can rewrite this result to give:

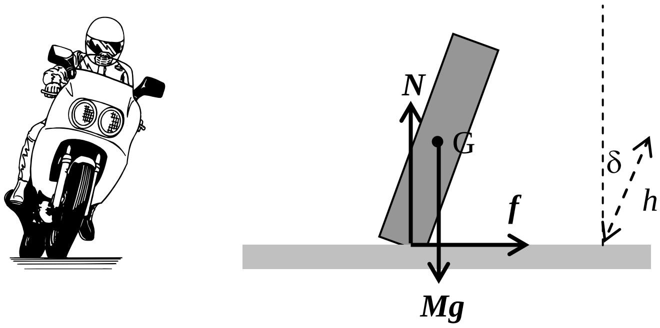
$$GM = \frac{4\pi^2 R^3}{\tau^2} = \frac{4\pi^2 (3.82 \times 10^8)^3}{(27.3 \times 24 \times 3600)^2} = 0.396 \times 10^{15}\ m^3\ s^{-2}$$

This value agrees well with that deduced for the acceleration of the apple, so we deduce that Newton's inverse square law of gravity can explain both the orbit of the moon and the motion of an apple as it falls off the tree.

7.9 The stability of a vehicle travelling in a circle

When a motor vehicle rounds a bend in a road, the centripetal acceleration is provided by the adhesion between the tyres and the road. If the adhesion is inadequate (e.g. due to ice or oil on the road, bald tyres etc.) then the vehicle will slide at a tangent to the curve and run off the road. There is a second stability criterion due to the turning moment of the friction about the centre of gravity. In the case of a motorcycle, the rider leans sideways into the bend, producing an equal and opposite turning moment. If a car were to take a bend at too high a speed, the turning moment due to the friction could cause the car to overturn. Let us examine these stability criteria in turn.

A motorcycle



Suppose a motorcyclist is rounding a bend with a radius of curvature *R* at a velocity *v.* The centripetal acceleration required is

$$a = v^2 / R$$

The maximum sideways friction that can be provided by the adhesion between the wheels and the road surface is

$$f = \mu_s N$$

where μ_s is the coefficient of static friction. If the mass of the rider plus machine is *M*, then

$$N = Mg$$

and

$$f = \mu_s N = \mu_s Mg$$

The greatest allowed acceleration is then

$$a = f / M = \mu_s g$$

so the maximum velocity at which the bend can be approached is when

$$a = v^2 / R = \mu_s g$$

giving

$$v_{\max} = \sqrt{\mu_s g R}$$

Now let us consider the effect of the turning moment of the friction. Suppose that the centre of gravity G of the motorcycle plus rider is a distance *h* above the ground and the rider tilts the bike at an angle δ to the vertical when rounding the bend. The stability condition is then that the turning moment about the centre of gravity G due to the frictional force *f* between the wheels and the road is balanced by the turning moment of the normal reaction force N. Referring to the figure above, we see that the condition is:

$$f\,h\cos\delta = N\,h\sin\delta \tag{i}$$

Now we can write

$$N = Mg$$

and

$$f = Mv^2/R$$

Inserting these results into (i) gives

$$Mv^2 h\cos\delta / R = Mgh\sin\delta$$

It follows that the angle δ must satisfy the condition

$$\tan\delta = \frac{v^2}{gR}$$

so that

$$\delta = \tan^{-1}\left(\frac{v^2}{gR}\right)$$

Question 7n

A motorcyclist goes around a bend of radius 30 m. If the coefficient of friction between the tyres and road is 0.32, calculate

(a) The maximum velocity at which the machine may negotiate the bend, and

(b) The required angle of inclination of the motorcyclist to the vertical.

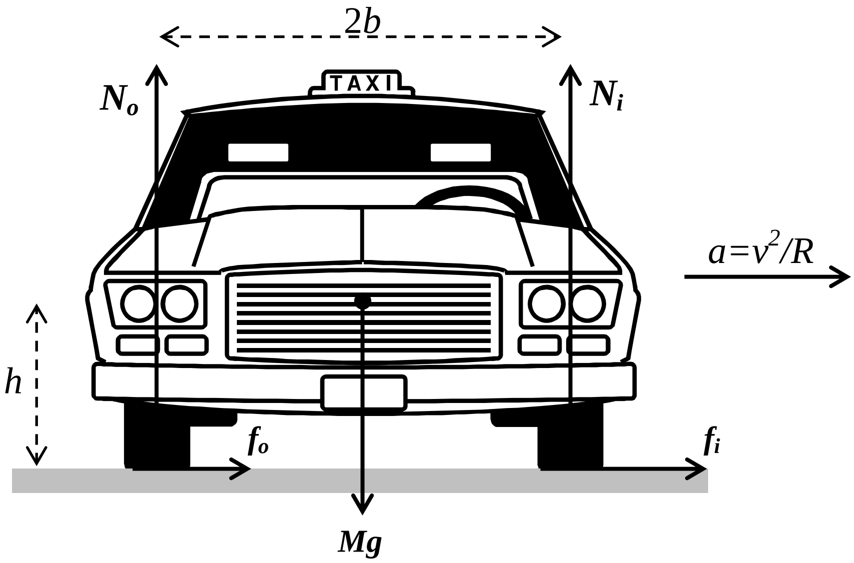
Solution: from the first stability condition

$$v_{\max} = \sqrt{\mu_s gR} = \sqrt{0.32\times 9.8\times 30} = 9.7ms^{-1}$$

or 34.9 km.h^{-1}. The angle is then

$$\delta = \tan^{-1}\left(\frac{v^2}{gR}\right) = \tan^{-1}\left(\frac{9.7^2}{9.8\times 30}\right) = 17.7^\circ$$

A car



Consider a car taking a bend of radius R at a velocity v. Suppose that the height of the centre of mass above the road is h and the lateral wheel base is $2b$. The centripetal acceleration required is

$$a = v^2 / R$$

This is provided by the friction between both pairs of wheels and the road.

In the sketch we have taken the total friction for the outer pair of wheels to be f_o and for the inner pair to be f_i. These are not equal, since the normal reaction forces N_o, N_i for the outer and inner pair of wheels are different. This is a consequence of the turning moment of the friction forces about the centre of mass of the car.

Resolving the forces vertically we have:

$$N_o + N_i = Mg \tag{i}$$

and the equation of motion for the car is

$$f_o + f_i = Mv^2 / R \tag{ii}$$

Let us now consider the turning moments about the centre of mass. The condition for stability is

$$(f_o + f_i)h = (N_o - N_i)b \tag{iii}$$

Substituting for $(f_o + f_i)$ from (ii) we find

$$N_o - N_i = \frac{Mv^2 h}{Rb} \tag{iv}$$

Adding and subtracting equations (i) and (iv) gives:

$$N_o = \frac{M}{2}\left[g + \frac{v^2 h}{Rb}\right]$$

and

$$N_i = \frac{M}{2}\left[g - \frac{v^2 h}{Rb}\right].$$

It can be seen that the normal reaction force on the outer wheels is increased, while the normal reaction force on the inner wheels is decreased. This fits with everyday experience (it is very noticeable in a car with soft suspension, like the old Citroen 2CV). The condition for the car to be on the point of overturning on the bend is that the normal reaction force on the inner wheels vanishes. From above this condition is

$$g = \frac{v^2 h}{Rb}$$

so the maximum velocity for rounding the bend without overturning the car is

$$v_{\max} = \sqrt{\frac{gRb}{h}}$$

The other possibility is that the adhesion between the wheels and the road may not be great enough, and the car would slide out of the bend. In this case both f_o and f_i would have to reach the limiting values $f_o = \mu_s N_o$ and $f_i = \mu_s N_i$. Equation (ii) then becomes

$$f_o + f_i = \mu_s (N_o + N_i) = \mu_s Mg = Mv^2 / R$$

The maximum velocity to avoid sliding is then

$$v_{\max} = \sqrt{\mu_s gR}$$

Question 7o

A car rounds a bend of radius 80 m at a speed of 50 km h^{-1}. The car's centre of gravity is midway between the wheels and is 0.6 m above the ground. The car's lateral wheelbase is 1.4m. Find the value of the coefficient of friction necessary to prevent side-slip at this speed. Show that the car will not have overturned.

Solution: the coefficient of friction to avoid side-slip is

$$\mu_s = \frac{v_{\max}^2}{gR} = \frac{(50/3.6)^2}{9.8 \times 80} = 0.246.$$

The velocity at which the car would overturn is

$$v_{\max} = \sqrt{\frac{gRb}{h}} = \sqrt{\frac{9.8 \times 80 \times 0.7}{0.6}} = 30.2 ms^{-1} \equiv 108.9\ kmh^{-1}$$

This is more than twice as fast as the actual speed of the car, so we conclude the car would not overturn at 50 km h^{-1}.

7.10 Centroids

We have already encountered the concept of the "centre of mass" of an object. The position of the centre of mass of a set of masses m_i located at points $\underline{r}_i$ is given by

$$\underline{r}_{CM} = \sum_i m_i \underline{r}_i \Big/ \sum_i m_i$$

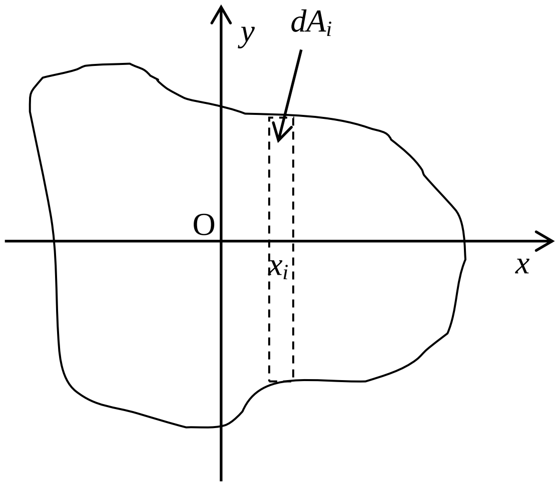
or

$$\underline{r}_{CM} = \frac{1}{M} \sum_i m_i \underline{r}_i$$

where M is the total mass. For a continuous mass distribution it is often convenient to replace the sum by an integral ($\int \rho \underline{r}\, dV$).

Below, we will deal specifically with calculating the centre of mass of laminae, i.e. bodies in the form of flat sheets of uniform density and thickness.

Consider the lamina shown in the diagram. Suppose we want to find the x-coordinate of the centre of mass. We may take the whole shape as made up from elements of area in the form of thin strips perpendicular to the x-axis, such as dA_i, at position x_i, as shown:



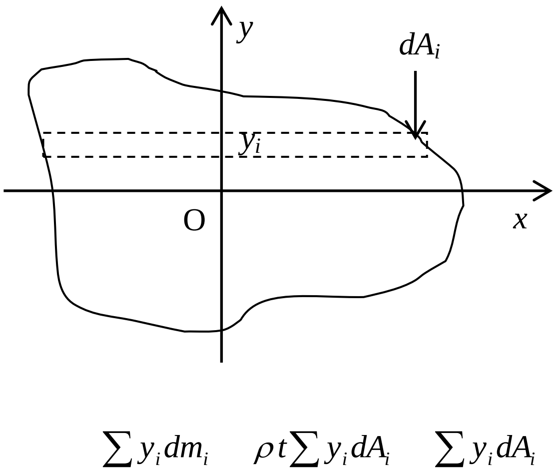
The mass of each element is

$$dm_i = \rho t\, dA_i$$

where ρ is the density of the material and t is the thickness of the lamina. The value of the x-coordinate of the centre of mass is given by

$$x_{CM} = \frac{\sum_i x_i dm_i}{\sum_i dm_i} = \frac{\rho t \sum_i x_i dA_i}{\rho t \sum_i dA_i} = \frac{\sum_i x_i dA_i}{\sum_i dA_i}$$

We can use the same method to find the y- coordinate of the centre of mass. In this case we take elementary strips normal to the y- axis:



It follows then that

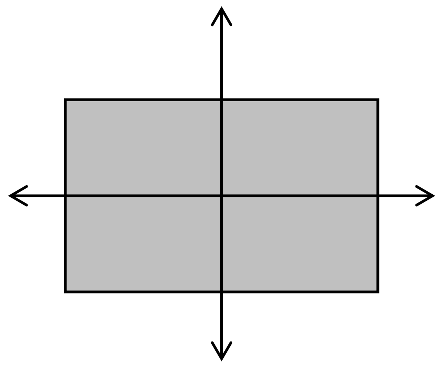
$$y_{CM} = \frac{\sum_i y_i dm_i}{\sum_i dm_i} = \frac{\rho t \sum_i y_i dA_i}{\rho t \sum_i dA_i} = \frac{\sum_i y_i dA_i}{\sum_i dA_i}$$

The centroid and the centre of mass

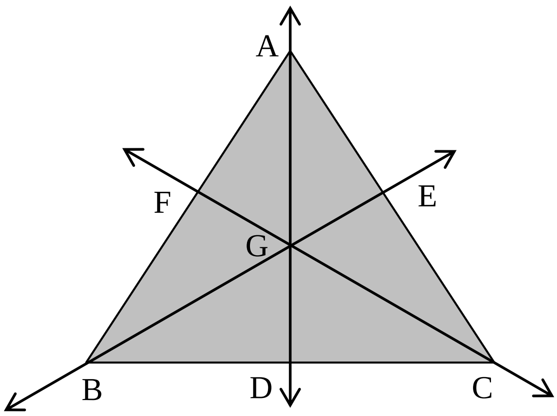
The centroid of an object is its geometric centre, relating to the way the volume (or area in the case of a lamina) is distributed. If the object has a uniform density, then the centre of mass will coincide with the centroid.

Use of symmetry to help determine the centroid or *com*

If a lamina has an axis of symmetry, then the centre of mass must lie on that axis. An obvious example is a rectangular lamina, as shown in the sketch. This has two symmetry axes at right angles to each other, parallel to the sides. If the lamina were reflected in the symmetry axes, it would map back onto itself. It follows that the centre of mass of a rectangular lamina most be where the two symmetry axes cross.



In the case of an equilateral triangle, there are three symmetry axes with angles of 60° between them. The centroid of the triangle is G, where the symmetry axes cross. Some simple trigonometry shows that AG = ⅔ AD, so the centroid is two-thirds of the way from each apex along the perpendicular bisector of the opposite side. This can be shown to be true for any triangle.



Centroids of composite laminae

Quite often we come across examples where a lamina can be broken down into a number of discrete elements, whose individual centroids are easy to find from symmetry.

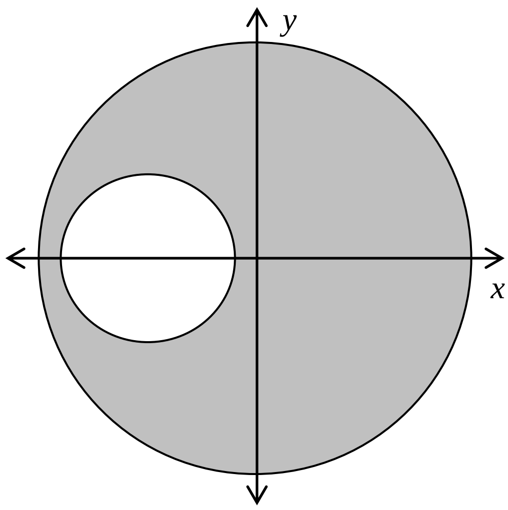
In this case we can find the centroid of the composite lamina simply by weighting the centroids of the individual components by their corresponding areas. Suppose the composite lamina is made up of N individual components whose areas are A_i and whose centroids have coordinates x_i and y_i.

Then the centre of mass (or centroid) of the whole has coordinates

$$x_{CM} = \frac{\sum_{i=1}^{N} x_i A_i}{\sum_{i=1}^{N} A_i} \quad \text{and} \quad y_{CM} = \frac{\sum_{i=1}^{N} y_i A_i}{\sum_{i=1}^{N} A_i}.$$

Question 7p

A circular hole of 50 mm diameter is cut in a circular disc of 120 mm diameter, the centre of the hole being 30 mm from the centre of the disc. Find the position of the centroid of the remainder of the disc.



Solution: we choose our origin to be the centre of the disc, so that before the hole is cut the centroid is at (0,0). By symmetry the y-coordinate of the centroid will still be at y = 0 after the hole is cut, since the x-axis is still a symmetry axis.

Let the remainder of the disc have area A_1 and the area of the hole be A_2. The total area of the disc before the hole is cut is

$$A_{tot} = \pi \times 60^2 = 11309.7mm^2 .$$

The area of the hole is $A_2 = \pi \times 25^2 = 1963.5mm^2$*, so the area of the remainder after the hole is cut is*

$$A_1 = A_{tot} - A_2 = 11309.7 - 1963.5 = 9346.2mm^2 .$$

Let the centroid of the hole be at $x_2 = -30mm$, and the centroid of remainder be at x_{com}. We want to determine x_{com}.

If we add the piece cut out of the disc, of area A_2, to the remainder we have the original disc, whose centroid is at $x = 0$. It follows therefore that

$$A_1 x_{com} + A_2 x_2 = A_{tot} \times 0 = 0$$

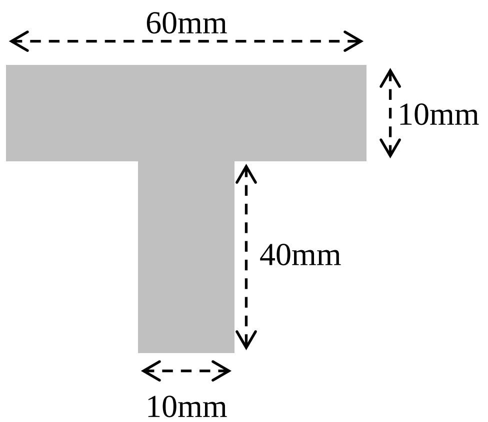
We then have

$$x_{com} = -\frac{A_2 x_2}{A_1} = -\frac{1963.5 \times (-30)}{9346.2} = 6.3mm$$

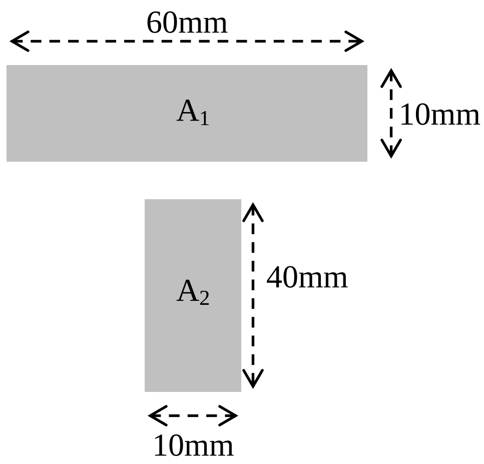
The centroid of the remainder therefore has coordinates $x = 6.3$ mm and $y = 0$.

Question 7q

Find the position of the centroid of the lamina shown:



Solution: this "T" shape is clearly made up of two rectangular laminae, as shown below:



By symmetry the centroid of the "T" must lie along the vertical centre line, 30mm from the left hand edge.

To find the vertical position of the centroid we break it down into the two rectangular laminae shown above. We can measure the positions of the centroids from the top edge of the "T". The cross-bar has area $A_1 = 60\times 10\ mm^2 = 600\ mm^2$*, and* $y_1 = -5mm$*, while the upright has area* $A_2 = 40\times 10\ mm^2 = 400\ mm^2$*, and* $y_2 = -(10+20)\ mm = -30\ mm$*.*

The total area is $(A_1+A_2) = 1000\ mm^2$*. It follows, therefore, that*

$$y_{com} = \frac{\sum_i y_i A_i}{\sum_i A_i} = \frac{(-5)\times 600 + (-30)\times 400}{1000} = \frac{-15000}{1000} = -15mm.$$

The centroid of the "T" shape has coordinates (30 mm, -15 mm) with respect to an origin at the top left hand corner of the "T".

This is a section of *Force, Motion and Energy*. It results from the work of several people over many years, with editing and additional writing by Martin Counihan.

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More information is given in the preface which forms the first file of this set.