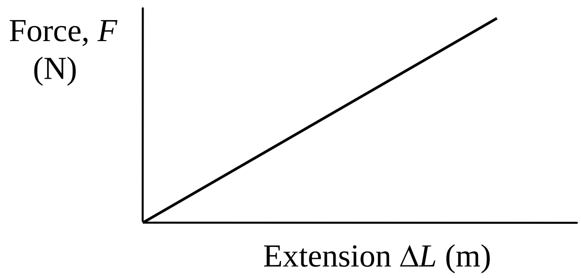
Section 2 Elasticity

One of the standard ways of measuring a weight is to use a calibrated spring balance. This relies on the fact that when a force is applied to a spring it stretches by an amount that is proportional to the magnitude of the force. When the force is removed the spring returns to its original length. This is a simple example of elastic behaviour. We are used to seeing elastic behaviour in rubber bands: they can increase in length by large factors (e.g. bungee cords) but will return to their original length when the force is removed. In fact all solids stretch when loaded, but the length changes are usually too small to be seen readily by the naked eye.

2.1 Hooke's Law, Tensile Stress and Young's Modulus

Suppose we secure the top end of a vertical wire of length L and cross-sectional area A. We then load the other end by hanging different masses on it. As long as the load is not too great we will find that the wire behaves elastically: the change in length of the wire ΔL is found to be proportional to the applied load. A graph of load (force F) against the extension ΔL gives a straight line:



This is an example of Hooke's Law, which states that the change in length of a material is proportional to the applied force. The stretching of the material sets up internal forces which balance the applied external force. Most solids obey Hooke's Law, at least for moderate loads. We can express Hooke's Law mathematically by:

$$F = k\,\Delta L, \tag{2.1}$$

where the constant k is sometimes called the mechanical stiffness and has the units of newtons per metre (N m^{-1}). If we are talking about a spring, the k is called the "spring constant".

The problem with equation (2.1) is that the value of the stiffness k depends on the dimensions of the wire, in particular its cross-sectional area and its length. For example if we took a wire of the same material, but with twice the cross sectional area, we would find it stretched only half as much. Similarly if we doubled the length of the wire, we would find that it stretched twice as much. We can take account of these factors by writing Hooke's Law in terms of the ***Stress*** and the ***Strain.***

Stress is defined as the force acting per unit of area; it is often given the symbol σ (Greek sigma) and its units (force/area) are N m^{-2}. Since a force per unit area is the same as a pressure, these are the same units as are used for pressure, and therefore we can call the units of stress "pascals" (Pa). So, we have $\sigma = F / A$.

The case above, of a loaded wire, is an example of **tensile stress**. In the case of heavy load on top of a vertical metal bar, we would have **compressive stress**.

Strain: this is defined as the fractional change in length of a material. Strain is often given the symbol ε (Greek epsilon) and is dimensionless, since it is just the ratio of two lengths:

$$\varepsilon = \frac{\Delta L}{L}$$

We can now transform Hooke's Law, eq. (2.1) into a relation between the stress and the strain, dividing both sides by the cross-sectional area:

$$\frac{F}{A} = \frac{k\,L}{A}\,\frac{\Delta L}{L} \qquad (2.2)$$

This equation is in the form:

Stress = Modulus x Strain

and can be written as

$$\sigma = E\,\varepsilon \qquad \text{or} \qquad \frac{F}{A} = E\,\frac{\Delta L}{L} \qquad (2.3)$$

Here the quantity E is called Young's modulus of elasticity. Here,

$$E = \frac{k\,L}{A}$$

However, the great advantage of writing Hooke's Law in the form of eq.(2.3) is that Young's modulus E, unlike the stiffness constant *k*, *does not depend on the dimensions of the material*. It is an intrinsic property that depends ultimately on the strength of the binding forces between the atoms of the material. Notice that since the strain is dimensionless, the units of *E*, the Young's modulus are the same as the units of stress, namely N m^{-2} or pascals (Pa).

The table below gives values of the Young's modulus for various metals and alloys, along with some other elastic moduli to be discussed later. It can be seen that typical values are of order 100 GPa (1 gigapascal = 10^9 pascals). In some engineering texts they quote values in kN mm^{-2}, which is the same as a GPa.

Material	***E*** (GPa)	***G*** (GPa)	**B** (GPa)	***Tensile Strength*** (Mpa)
Aluminium	71	26	74	150-450
Copper	130	48	138	300-500
Iron	211	82	170	400-600
Lead	17	5.5	46	10-15
Tin	45	18	58	100-150
Brass	37	37	112	350-550
Mild steel	212	82	169	1000-1200

Question 2a

A rectangular bar whose cross section has the dimensions 5 mm x 20 mm supports a mass of 500 kg. Calculate the stress in the bar.

Solution: the force on the bar is

$$F = mg = 500 \times 9.81\ N = 4905\ N.$$

The cross-sectional area of the bar is

$$A = 5 \times 20\ mm^2 = 10^{-4}\ m^2$$

So the stress is $F/A = 4.9 \times 10^7\ Nm^{-2} = 49\ MPa.$

Question 2b

A wire of length 2 m is acted on by a tensile force. What is the strain in the wire if it extends by 0.25 mm?

Solution: the strain is

$$\varepsilon = \frac{\Delta L}{L} \quad = \quad \frac{0.25 \times 10^{-3}}{2} \quad = \quad 1.25 \ \text{x} \ 10^{-4}$$

Question 2c

A steel rod 1 m long is subjected to a tensile stress of 100 MPa. The extension of the rod due to the stress is measured to be 0.5 mm. Find the value of Young's Modulus for the material of which the rod is made.

Solution: the strain is

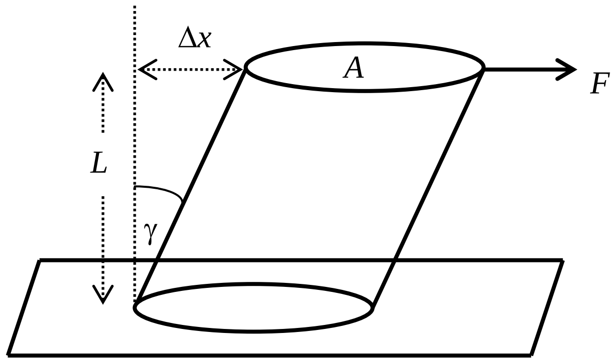
$$\varepsilon = \frac{\Delta L}{L} = \frac{0.5 \times 10^{-3}}{1} = 5 \ \text{x} \ 10^{-4}$$

so Young's modulus is:

$$E = \frac{\sigma}{\varepsilon} = \frac{Stress}{Strain} = \frac{100 \times 10^{6}}{5 \times 10^{-4}} = 20 \times 10^{10}\ \text{Pa} = 200\ \text{GPa}$$

2.2 Shear Stress and the Shear Modulus

Like the kind of stress considered above, a **shear stress** is also a force per unit area, but in this case the direction of the applied force lies in the plane of the area rather than perpendicular to it:



The figure shows the response of a material to a shear stress. Imagine a cylinder of length *L* and cross sectional area *A.* One end is firmly attached to a surface, the other end is subjected to a force *F* parallel to the face. The shear stress is given the symbol τ (Greek tau):

$$\text{Shear Stress} = \frac{\text{Applied force}}{\text{Area Resisting Force}}$$

$$\text{in symbols} \quad \tau = \frac{\text{F}}{\text{A}}$$

The material responds to the shear stress by deforming sideways through a small distance Δx.

We define the shear strain as the ratio between the relative displacement (Δx) between any two planes and the perpendicular distance (L) separating the two planes.

In the example illustrated above, the shear strain is $\frac{\Delta x}{L} = \tan\gamma$, where γ is the angle of deformation.

If the shear stress is not too large, we would expect the shear strain to be proportional to the shear stress. We can therefore define another type of modulus, called the "modulus of rigidity", or the "shear modulus":

$$\text{shear stress} = \text{shear modulus} \times \text{shear strain}$$

or,

$$\tau = \frac{F}{A} = G \tan\gamma$$

For a small shear strain, $\tan\gamma \approx \gamma$ (when γ is measured in radians) and so we may approximate the equation to:

$$\boxed{\tau = \frac{F}{A} = G\gamma}$$

Shear Stress

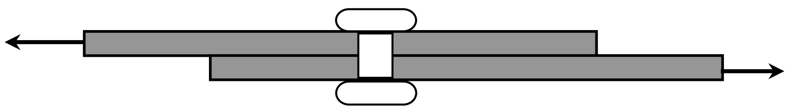
The calculation of shear stress is problem-dependent. It is necessary to identify carefully the area that resists the shear force. In all cases shear stress is calculated from this expression:

$$\text{Shear Stress} = \frac{\text{Applied force}}{\text{Area Resisting Force}}$$

$$\text{or in symbols:} \quad \tau = \frac{F}{A}$$

Two typical cases are illustrated below.

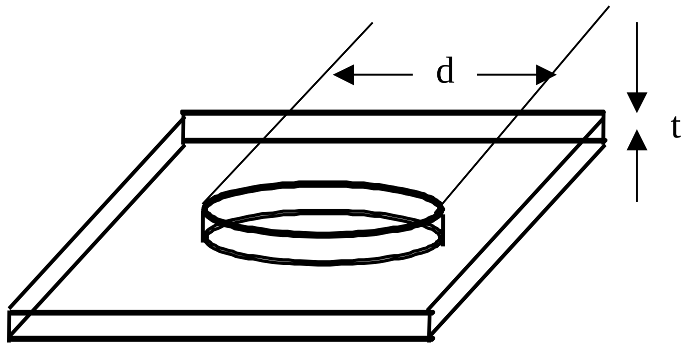
A Riveted Joint: **F**



Let the diameter of the rivet be *d*. The shear stress is then calculated from:

$$\tau = \frac{F}{A} \quad \text{where} \quad A = \frac{\pi d^2}{4} \quad \text{(cross-sectional area)}$$

Punching a hole in a plate:



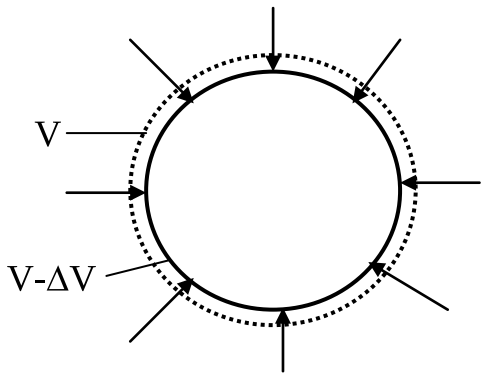
Let the plate thickness be t and the diameter of the hole be d. If the shear force required is F then the shear stress is calculated from

$$\tau = \frac{F}{A},$$

where $A = \pi d\, t$.

2.3 Hydraulic Stress and the Bulk Modulus

Imagine a body subject to a uniform compression over all its surface:



This sort of situation would occur, for example, if the body were taken down to the sea bed. The weight of water above creates a high pressure, known as hydrostatic pressure, which acts in all directions.

We can calculate the magnitude of the pressure as follows: imagine a column of water of unit cross sectional area ($1 m^2$) and height h. Then the volume of water in this column will be $(1 \times h)\ m^3 = h\ m^3$.

The mass of water in the column is then = volume x density = $h\rho$, and is in kilograms, where ρ is the density of water (in kilograms per cubic metre).

The weight of this column is then equal to $h\rho g$ (in newtons) where g is the acceleration due to gravity (9.81 m s^{-2}).

This is then the force acting over a unit area ($1m^2$), so the pressure will be $h\rho g/(1\ metre\ squared) = h\rho g$ (N m^{-2} or Pa).

Suppose we want to know the pressure at a depth of 1 km. Then, taking ρ = 1000 kg m^{-3} and $g \approx 10$ ms^{-2}, we have:

$$pressure = 10^3 \times 10^3 \times 10 = 10^7\ Pa$$

This is about 100 times normal atmospheric pressure.

When subject to hydrostatic pressure, or hydraulic stress, all materials decrease in volume. The hydraulic stress is the force acting per unit area of surface, which is just the pressure p. The hydraulic strain is the ratio of the change in volume to the original volume of the body, i.e.

$$\text{hydraulic strain} = \frac{\Delta V}{V}$$

The constant of proportionality between the hydraulic stress and the hydraulic strain is called the bulk modulus, B, so we have:

$$\text{hydraulic stress} = \text{bulk modulus} \times \text{hydraulic strain}$$

or

$$\frac{F}{A} = p = B\frac{\Delta V}{V}$$

Question 2d

The bulk modulus of water is 2.2 GPa. Calculate the fractional change in the volume of water at the bottom of the Pacific Ocean, at a depth of 4000m.

Solution: from what was written above, the pressure at a depth of 4000 m will be 4×10^7 *Pa. The hydraulic strain* $\Delta V/V$ *for the water is just the pressure divided by the bulk modulus, so that*

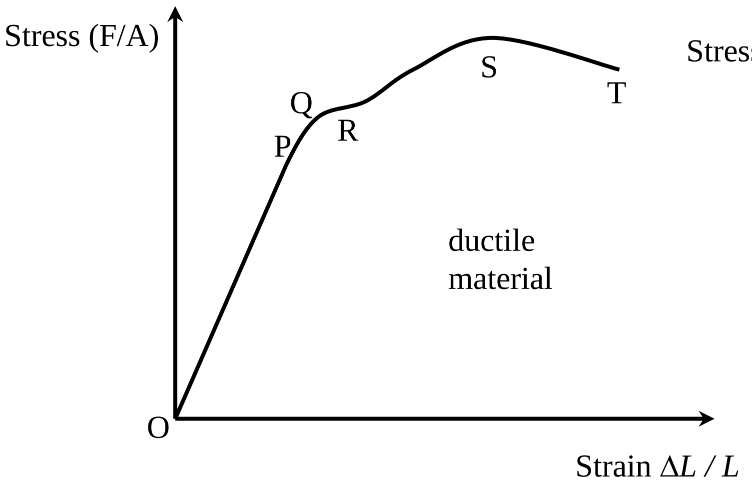
$$\Delta V/V = \frac{4\times10^7}{2.2\times10^9} = 0.018 \qquad \textit{(or 1.8\%)}$$

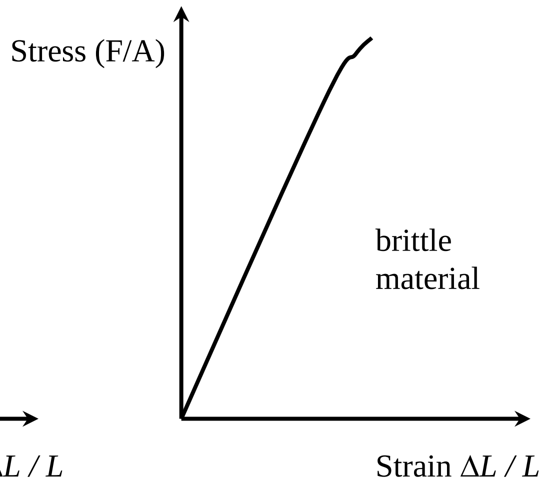
The bulk moduli for some common metals are listed in the table given earlier. The values for solids are generally higher than those of liquids, for example B for iron is 170 GPa. This means that the fractional volume change for iron would be only 0.025% at the bottom of the Pacific Ocean.

2.4 Tensile Testing of Materials

So far we have considered relatively small stresses and strains, so that the elastic limit is not exceeded. But how do we know when we are working within the elastic limit? The mechanical properties of materials are strongly dependent on their previous mechanical history, by their chemical composition (if we are dealing with an alloy) and by the heat treatment they have received. For example, to make a metal ductile it needs to be well annealed and free of precipitates. We can increase the strength of metals by hardening techniques (e.g. cold-drawing or rolling) and by precipitating secondary phases. Hardening tends to tangle up dislocations, while precipitates act as pinning centres for dislocations. In both cases the dislocations are prevented from moving. As a result the material is brittle but strong.

Tensile testing allows us to establish the elastic limit for a particular material, but also shows how a sample behaves when the elastic limit is exceeded. The figures following show the stress/strain plot typical of the tensile test curves for a ductile material and a brittle material.





To find the elastic limit, increasing tensile forces are applied to a regularly shaped sample of material and the resulting strains are measured. Below the elastic limit (the section of curve labelled OP in the figure) the strains are proportional to the applied stresses. The point P is called the proportionality limit. Above this point the curve departs from a straight line, but below the elastic limit at Q the strains disappear when the stresses are removed. Going above Q leads to a permanent strain when the stresses are removed. R is called the yield point: here the specimen extends with little or no increase in load. The point of maximum stress S defines the ultimate tensile stress, or "tensile strength" of the material:

$$\text{tensile strength} \quad = \quad \text{maximum tensile stress}$$

At this point, the extension becomes localised, and the specimen acquires a "waist". At point T the specimen fractures.

In actual samples, the points P, Q and R may coincide or may be difficult to distinguish. Note that for a brittle material the sample fails before or near to the yield point.

Besides tensile strength, compressive strength and shear strength may be similarly defined.

In engineering, a material should never be put into a situation where it might be subject to a stress comparable to its strength. To avoid this happening, a suitable safety factor is chosen at the design stage. **The permissible stress is then obtained by dividing the strength by the safety factor.**

This is a section of *Force, Motion and Energy*. It results from the work of several people over many years, with editing and additional writing by Martin Counihan.

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More information is given in the preface which forms the first file of this set.