Section 5: Work, Power and Energy

5.1 Work and power

If a constant force F acts on an object which moves a distance s in the direction of application of the force, the work done is the product of the force times the displacement:

$$\boxed{Work = Force \times Displacement}$$

or

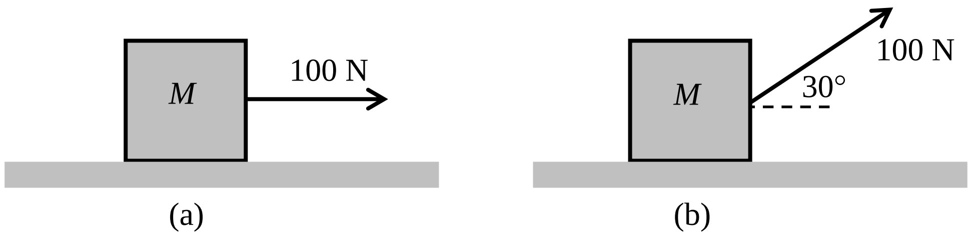
$$W = Fs$$

The unit of work must obviously be the newton-metres (N m) but this unit has its own special name, the joule (J). Although the word "work" is often used in the context of movement through force, the more general term is "energy".

Note that it is the *displacement in the direction of the force* that comes into the definition of work. In a more general case, we should treat the force as a vector quantity $\underline{F}$ and take the displacement as small distance $d\underline{r}$, also a vector. Then the element of work done can be written as a scalar product:

$$dW \quad = \quad \underline{F} \cdot d\underline{r}$$

This means that it is only the component of the force in the direction of the displacement that does work. For example, consider the two cases shown below:

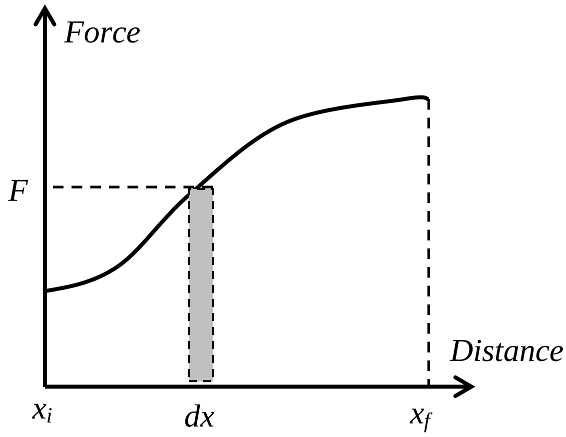


In (a), on the left, a crate of mass M is pulled across a smooth floor by a force of 100 N for a distance of, say, 20 m. The work done is $W = Fs$ = 100 × 20 = 2000 J. For the situation in (b), on the right, the work done is $W = (F \times \cos(\theta)) \times s$ = 100 cos(30°) × 20 = 1732.1 J.

If the force changes with distance we have to add up all the elements of work done along the path on which the body moves. We can write this as an integral:

$$W_{i \to f} \quad = \quad \int_{\underline{r}_i}^{\underline{r}_f} \underline{F} \cdot d\underline{r}$$

We can interpret the integral as the area under a graph of force against distance, as shown below:



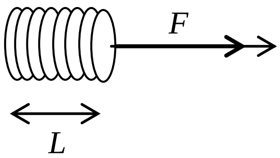
A Force-distance graph for a variable force in one dimension. The element of work dW when the object moves a distance dx is the area under the graph (shaded)

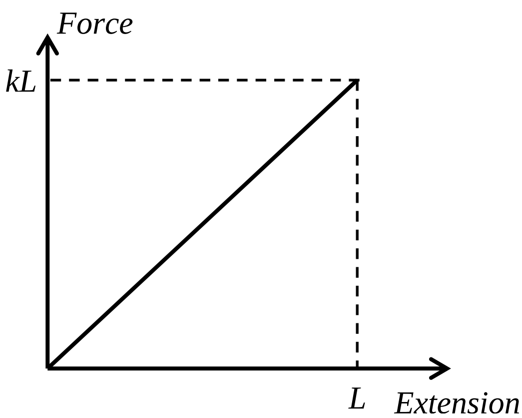
$$dW = F\,dx$$

The total work done in moving the body from position x_i to position x_f is the total area under the curve, given by the integral

$$W_{i\to f} \;=\; \int_{x_i}^{x_f} F\,dx$$

Work Done in Stretching or Compressing a Spring





The force required to stretch a spring is proportional to its extension (Hooke's Law): in magnitude,

$$F = k\,x$$

where *k* is the spring constant. The work done in stretching a spring by a distance *L* is therefore the area under the graph of F against x, the diagonal straight line in the diagram. This is just the area of a triangle of base *L* and height *F=kL*. This is just

$$W = \tfrac{1}{2}L \times kL = \tfrac{1}{2}kL^2$$

We would get the same answer by using an integral:

$$W \quad = \quad \int_0^L F(x)\,dx \quad = \quad \int_0^L kx dx \quad = \quad [½\,kx^2]_0^L \quad = \quad ½\,kL^2$$

For a spring obeying Hooke's Law, the work done in *compressing* the spring by a distance *L* is the same as the work done in stretching it by the same amount. Although in compression the sign of the force is negative (since *x,* the extension, is negative), as we compress it the increment *dx* is also negative, so the work is still positive:

$$W \quad = \quad \int_0^{-L} F(x)dx \quad = \quad \int_0^{-L} (-kx)(-dx) \quad = \quad [½\,kx^2]_0^{-L} \quad = \quad ½\,kL^2$$

Power

Power is the rate at which work is done:

$$\boxed{P = \frac{Work}{Time} = \frac{dW}{dt}}$$

The units of power are joules per second (J s^{-1}) or watts.

Using the expression for the element of work

$$dW \quad = \quad \underline{F} \cdot d\underline{r}$$

The power can be written:

$$P \quad = \quad \frac{dW}{dt} \quad = \quad \underline{F} \cdot \frac{d\underline{r}}{dt} \quad = \quad \underline{F} \cdot \underline{v}$$

So the power is the product of the force and the velocity:

$$\boxed{Power = Force \times Velocity}$$

Question 5a

A winch raises a load of 0.2 tonnes through a height of 5 m in 10 seconds at a constant rate. Calculate the power of the winch motor.

Solution: the tension T *in the winch cable exactly balances the weight* Mg *of the load, since the load is not accelerating. The power is therefore* $T \times v = Mgv = 0.2 \times 10^3 \times 9.8 \times (5/10) = 980$ *watts.*

5.2 Work and kinetic energy

Suppose a body of mass m is travelling initially at a velocity u. If a constant force F acts on the body over a distance s, the body will accelerate at a rate $a = F/m$ to a final velocity v, where

$$v^2 = u^2 + 2as\,.$$

How does this relate to the work done? The work done is:

$$W \;=\; Fs \;=\; mas \;=\; \tfrac{1}{2}\, m(v^2 - v^2)$$

The right hand side of this equation is just the change in the kinetic energy of the body.

$$\boxed{Kinetic\ Energy = \tfrac{1}{2} mv^2}$$

If all the work done on a body goes into accelerating it, we have:

$$\boxed{Work\ done = Change\ in\ Kinetic\ Energy}$$

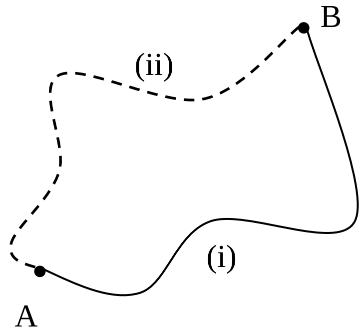
5.3 Work and potential energy

We have seen that we have to do work in order to accelerate a body over a certain distance. However, there are also situations where just changing the position of a body requires work, even though the body acquires no velocity. For example if I lift a block of metal of mass m from the floor in order to place it on a table at a height h above the floor, I have to do work against the force of gravity acting on the block.

Another example would be the work done in moving a mass attached to a spring. In this case we are doing work against the elastic forces resulting from the deformation of the material of the spring.

Conservative forces

Suppose we want to move a body from the point A (with position vector $\underline{r}_i$) to the point B (position $\underline{r}_f$) as illustrated. The work done in such a process would be the sum of all the elements of work :



$$W_{i\to f} \;=\; \int_{\underline{r}_i}^{\underline{r}_f} \underline{F}\cdot d\underline{r}$$

The answer we get when we calculate this integral might depend on which path we choose to take: for example the work done along path (i) in the diagram might be different from the work done along path (ii). There is, however, an important class of forces for which the work done is *independent of the path taken.* Such forces are called *conservative forces*.

For a conservative force we can define the *potential energy* at a point, *U*, such that the difference between the potential energies at two points is:

$$U(\underline{r}_f) \;-\; U(\underline{r}_i) \;=\; -\int_{\underline{r}_i}^{\underline{r}_f} \underline{F}\cdot d\underline{r} \;=\; -\; W_{i\to j}$$

Note that only <u>differences</u> in potential energy are defined, so we are free to measure the potential energy starting from any convenient point. Suppose we increase the potential energy of a body by doing work against a force in making a displacement of the body. Then when the body is released, the force is free to do work, for example by accelerating the body and increasing its kinetic energy. The potential energy refers to the “potential” for doing work – hence the term

Gravitational potential energy

The force acting on a body of mass *m* due to the gravitational attraction of the Earth is

$$\underline{F} \;=\; m\underline{g} \;=\; -mg\underline{\hat{z}}$$

where $\hat{\underline{z}}$ is a unit vector, vertically upward. Since the force acts vertically downwards, it costs no effort to move a body in the *x* or *y* directions, so the potential energy does not depend on *x* or *y*. We now work out the difference in potential energy for a body of mass *m* when it is lifted a height *h* above the Earth's surface:

$$U(h) - U(0) = -\int_0^h F_z dz = -\int_0^h (-mg)dz = mgh$$

We choose the zero of the potential energy to be when $z = 0$, so that $U(0) = 0$. Then the gravitational potential energy is:

$$\boxed{U(h) = mgh}.$$

The gravitational force is clearly conservative, since the work done by gravity when a body moves along a path only depends on the change in height between the starting point and the end point of the path.

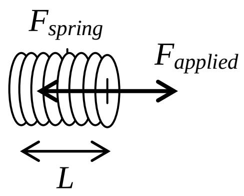
When we lift a mass through a height *h* we have to apply a force, equal and opposite to the downward gravitational force. The work done is therefore:

$$W = \int_0^h mg\,dz = mgh = U(h),$$

that is, the work done *on* the system is equal to the change in the potential energy of the system.

Potential energy of a spring

To stretch a spring we have to apply a force equal and opposite to the force exerted by the spring $F_{spring} = -kx$. The potential energy of a spring stretched through distance L is therefore



$$U(L) - U(0) = -\int_0^L F_{spring}(x)dx = -\int_0^L (-kx)dx\ [\tfrac{1}{2}kx^2]_0^L = \tfrac{1}{2}kL^2$$

Again we may take the potential energy to be zero when the spring is unstretched, so that $U(0)=0$. The potential energy of a stretched (or compressed) spring is then

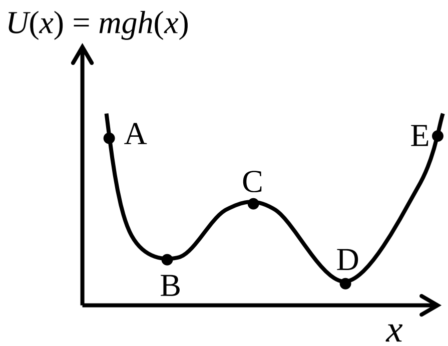
$$U(L)=\tfrac{1}{2}kL^2.$$

Note from what we showed earlier, that the work done *on* a spring by the applied force, in stretching the spring, is equal to the increase in the potential energy of the spring:

$$W = \int_0^L F_{applied}(x)dx = \int_0^L kxdx = [½\,kx^2]_0^L = ½\,kL^2 = U(L)$$

5.4 Conservation of mechanical energy

Suppose we have a body moving in one dimension, subject to only conservative forces. A simple example would be a bead sliding on a wire, with no friction. The wire is bent into the shape $h(x)$ shown in the sketch, where h is the height above the floor.



Then the potential energy of the bead is just proportional to the height:

$$U(x)=mgh(x).$$

If the bead is held initially at point A, its kinetic energy (KE) is zero but its potential energy (PE) is high. When released, the bead slides down the wire, losing PE, but gaining KE. The bead continues to accelerate until point B, where it starts to gain PE again at the expense of KE. After point C, it speeds up again as it starts to fall, reaching its maximum speed at the lowest point D. It then slows to a stop at point E, at the same height as point A, when all its KE is converted into PE again. If allowed to continue, the bead would continue to oscillate backwards and forwards between points A and E.

In this case the conservative force is gravity. As gravity does work W on the bead, its KE increases:

$$W = \Delta(KE),$$

but the work done by gravity is just the decrease in the potential energy:

$$W = -\Delta U = -\Delta(PE).$$

It follows that

$$\Delta(KE) = -\Delta(PE),$$

or equivalently

$$\Delta(KE) + \Delta(PE) = 0.$$

We can define the mechanical energy E of a body as the sum of the kinetic energy and the potential energy of the body. The above result may now be phrased as the *principle of conservation of mechanical energy*:

$$\Delta E = \Delta(KE) + \Delta(PE) = 0$$

$$\boxed{\begin{array}{l} \textit{For a system subject to conservative forces,} \\ \textit{the total mechanical energy cannot change} \end{array}}.$$

For motion in one dimension the mechanical energy may be written

$$E = \tfrac{1}{2}mv^2 + U(x).$$

Question 5b

A mass m is lifted from the floor through a height h and then released. What is its velocity when it strikes the floor?

Solution: use the principle of the conservation of energy. By lifting the mass through a height h its potential energy is increased by mgh. When the mass is released, the potential energy is converted into kinetic energy, so that by the point at which the mass strikes the floor its kinetic energy is

$$\tfrac{1}{2}mv^2 = mgh.$$

It follows that

$$v = \sqrt{2gh}.$$

This is just the same answer we would get by applying the equation for uniformly accelerated motion to this problem: $v^2 = u^2 + 2as$, with $u=0$, $s = -h$ and $a = -g$.

Question 5c

A block of mass 2 kg is dropped from a height of 40 cm onto a spring with stiffness 1960 N m^{-1}. What is the maximum compression of the spring, in bringing the mass to a halt?

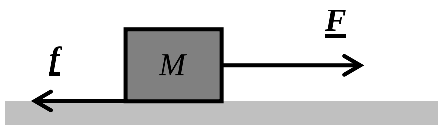
Solution: the gravitational potential energy lost by the block is converted ultimately into the potential energy of the compressed spring. So,

$$mgh \;=\; \tfrac{1}{2}\,k\,L^2$$

Putting in the numbers gives L = 89 mm.

5.5 Work and energy in the presence of friction

Consider the example in the sketch, where a heavy block of mass M is pulled across a rough floor by a force $\underline{F}$. The frictional force is $\underline{f}$ and the coefficient of kinetic friction is μ_k.



The net force acting on the block is $\underline{F} - \underline{f}$ and this produces an acceleration a:

$$F - f = ma$$

The work done by the force $\underline{F}$ in dragging the block a distance d is Fd.

$$W = Fd = mad + fd$$

Using $v^2 = u^2 + 2ad$, then $ad = \frac{1}{2}(v^2 - u^2)$, we can rewrite the work done as:

$$W \;=\; Fd \;=\; mad \;+\; fd \;=\; \tfrac{1}{2}\,m(v^2 - u^2) \;+\; fd \;=\; \Delta(KE) \;+\; w_f$$

From this result we see that some of this work is expended in increasing the kinetic energy of the block, the rest is the work w_f required to overcome the frictional force $\underline{f}$.

The forces involved in friction are, of course, non-conservative since if we reverse the path we still have to expend energy to pull the block back again. The work done against friction is dissipated in the form of heat and noise generated at the contact between the block and the floor. Clearly the mechanical energy is not conserved here.

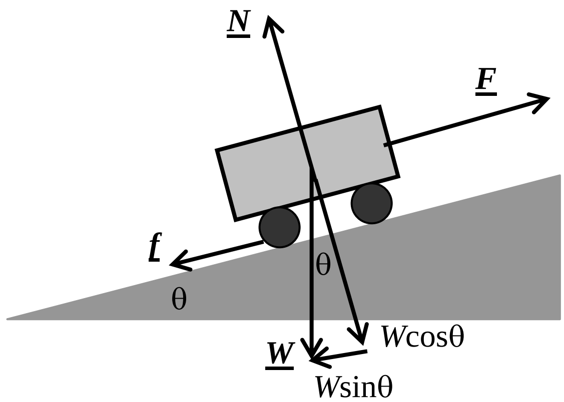
However once we realise that heat is just another form of energy, we see that the work done on the system by an external force (F in this case) is equal to the change in the *total* energy of the system, when we include *all* forms of energy in our account. For an isolated system this leads to a broader principle, the *conservation of energy*:

$$\boxed{\textit{The total energy of an isolated system cannot change}}.$$

While this is one of the basic principles of physics, it is not particularly useful for solving problems in mechanics, unless the forces are conservative, when it reduces to the principle of conservation of mechanical energy.

Question 5d

A truck of weight 600 kN is driven up a slope, inclined at 15° to the horizontal, at a constant speed of 2 m s^{-1}. There is a constant resistance to motion of 200kN. Calculate (a) the magnitude of the traction force required and (b) the work done in driving a distance of 150 m up the slope.



Solution: since the truck moves at a constant velocity, the truck is in equilibrium, i.e. there is no net force acting on it. Resolving the forces parallel to the slope:

$$F = f + W\sin\theta$$

so that the traction force is

F = 200 x 103 + 600 x sin(15°)

= 355 kN

The work done is then

$$W = Fd = 355\times10^3\times150 = 53.29MJ$$

Note that the work done by the traction force is made up of

(i) the work done against friction:

$$w_f = fd = 200\times10^3\times150 = 30MJ$$

(ii) the work done in increasing the potential energy of the truck,

$$mgh = mgs \times \sin\theta = 600 \times 10^3 \times 150 \times 0.2588 = 23.29MJ$$

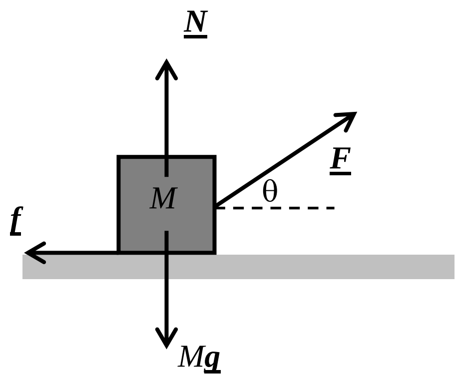
giving a total of $53.29MJ$ *. There is no increase in the kinetic energy here since the truck travels at a constant speed.*

Question 5e

A heavy crate of mass 200 kg is to pulled across a rough floor using a constant force of 1200 newtons. The coefficient of kinetic friction between the crate and the floor $\mu_k = 0.6$. Calculate the maximum possible acceleration of the crate and the time taken to drag the crate 10m.

Solution: the forces acting are shown in the diagram. The friction force is f, and N is the normal reaction force. Resolving the forces normal to the floor, we have:

$$N + F\sin\theta = Mg \qquad (i)$$



The net horizontal force is

$$F\cos\theta - f$$

which produces an acceleration a. The equation of motion for the crate is therefore

$$F\cos\theta - f = Ma \qquad (ii)$$

We can write the friction force in terms of the normal reaction force:

$$f = \mu_k N = \mu_k (Mg - F\sin\theta)$$

(iii)

In (iii) we have substituted the value of N from (i). Substituting for f in (ii) gives

$$Ma = F\cos\theta - f = F\cos\theta - \mu_k (Mg - F\sin\theta)$$

This gives the acceleration as

$$a = (F/M)\cos\theta - \mu_k [g - (F/M)\sin\theta]$$

(iv)

We now have to find the value of θ that maximises the acceleration. To do this we use a bit of calculus: the condition that a is a maximum is that

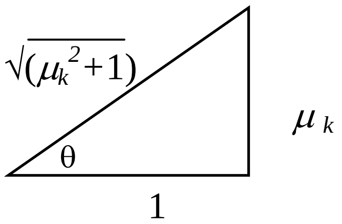
$$\frac{da}{d\theta} = 0$$

We find:

$$\frac{d\alpha}{d\theta} = \frac{F}{M}(-\sin\theta + \mu_k \cos\theta) = 0$$

giving $\tan\theta = \mu_k$. *So the optimal angle to apply the force to the crate is*

$$\theta = \tan^{-1}(\mu_k)$$



We see from the triangle that for this value of θ,

$$\sin\theta = \frac{\mu_k}{\sqrt{\mu_k^2+1}} \quad \textit{and} \quad \cos\theta = \frac{1}{\sqrt{\mu_k^2+1}}.$$

We can substitute these values into equation (iv):

$$a_{\max} = \frac{F}{M}\left[\frac{1}{\sqrt{\mu_k^2+1}} + \frac{\mu_k^2}{\sqrt{\mu_k^2+1}}\right] - \mu_k g = \frac{F}{M}\sqrt{\mu_k^2+1} - \mu_k g$$

Putting the numbers in we find:

$$a_{\max} = \frac{F}{M}\sqrt{\mu_k^2+1} - \mu_k g = \frac{1200}{200}\times\sqrt{1.36} - 0.6\times 9.8 = 1.117 ms^{-2}$$

Using $s = ut + \frac{1}{2}at^2$*, the time taken to drag the crate 10 m is*

$$t_{\min} = \sqrt{\frac{2S}{a}} = \sqrt{\frac{2\times 10}{1.117}} = 4.23\text{sec}$$

Note that from (iv) if we had applied the force parallel to the floor, the acceleration would be about ten times smaller: for $\theta = 0$

$$a = \frac{F}{M} - \mu_k g = \frac{1200}{200} - 0.6\times 9.8 = 0.12\ m s^{-2}.$$

In this case it would take 12.9 seconds to drag the crate 10 m.

In the absence of friction the acceleration would be

$$a = \frac{F}{M} = \frac{1200}{200} = 6\ m s^{-2}.$$

5.6 Energy in simple harmonic motion

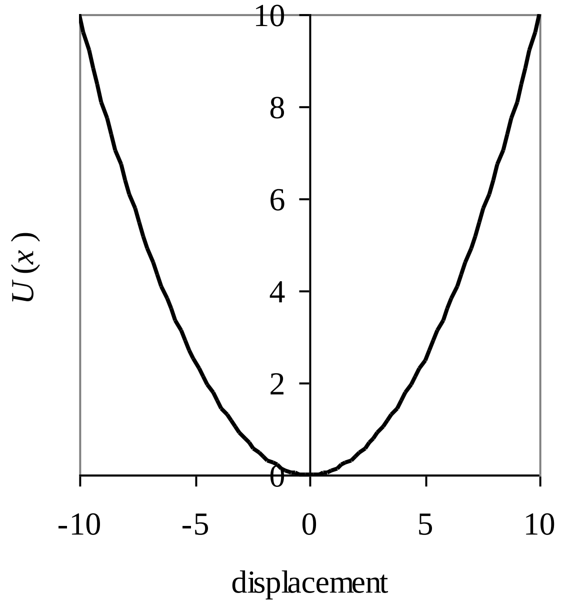
Simple harmonic motion (SHM) often occurs in mechanical systems when they are displaced from equilibrium. For example consider a mass hanging on an elastic string. Its weight has stretched the spring until the mass is in equilibrium between the weight *mg* acting downwards and tension *T* in the string acting upwards. Suppose we now pull the mass down a small distance and release it. The extra tension in the string will accelerate the mass upwards, then it will oscillate up and down at an angular frequency ω, executing SHM. Other examples include the motion of a simple pendulum, the vibrations of a mass on the end of a cantilever, etc. All of these systems are governed by a restoring force that is proportional to the displacement of the system:

$$F = -k\,x$$

Correspondingly the potential energy of the system is quadratic in the displacement, as for the elastic spring discussed in the previous section:

$$U(x) = \tfrac{1}{2} k x^2 .$$

The form of $U(x)$ is shown here:



The mechanical energy is the sum of the kinetic energy and the potential energy:

$$E = \tfrac{1}{2} m v^2 + \tfrac{1}{2} k x^2$$

Since the system is isolated, the total energy is constant. This means that if the potential energy increases, the kinetic energy must decrease, and *vice versa*. To study the time dependence of the problem, suppose that the displacement changes with time like a cosine:

$$x = A\cos(\omega t)$$

where *A* is the amplitude and ω is the angular frequency of the oscillation. It follows that the velocity is:

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t)$$

The total mechanical energy is then

$$E = ½\, m\, [-A\omega \sin(\omega t)]^2 + ½\, k [A \cos(\omega t)]^2$$

The only way the total energy can be made a constant (i.e. independent of time t) is for the coefficients of $sin^2(\omega t)$ and of $cos^2(\omega t)$ to be identical, since

$$sin^2(\omega t) + cos^2(\omega t) = 1$$

In this case we would have

$$\tfrac{1}{2} m A^2 \omega^2 = \tfrac{1}{2} k A^2$$

Hence the natural frequency of oscillation is:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

This is the standard result for SHM, and the total energy is:

$$E = \tfrac{1}{2} k A^2 = \tfrac{1}{2} m \omega^2 A^2$$

The illustration below shows the time dependence of kinetic energy and potential energy in SHM. Note that the total energy KE+PE is constant.