Section 6: Collisions

For an isolated system, the total linear momentum is a conserved quantity. If two bodies of mass m_1 and m_2 interact *only with each other* then

$$m_1\underline{v}_1 + m_2\underline{v}_2 = constant$$

This principle might seem to have rather limited usefulness, since it is not very often that we deal with isolated systems of particles. For example in a car crash, both cars are in contact with the road and probably at least one driver will have applied the brakes. Both cars are therefore subject to frictional forces with the road, and so the "system" (comprising the two cars) is not strictly isolated. However the impulse involved when two cars collide involves a large force acting for a very short time. In such a short time the change in momentum of the cars due to the external frictional forces will be negligible, compared to the change in momentum due to the impulse of the collision. It follows that we can usually make use of the principle of conservation of linear momentum even when the system is not isolated.

Question 6a

A car *P* of mass 1200 kg travels along a level road at a speed of 8 m s^{-1}. The driver notices that car *Q* ahead has stopped and immediately applies the brakes, causing the wheels to lock. The car *P* slides a distance of 10 metres before it hits car *Q* at a speed of 2 m s^{-1}. Car *Q* has a mass of 800kg. Immediately after impact car *P* moves with a speed of 1 m s^{-1}. Suppose the collision has a duration of 0.1 seconds: analyse what happens in the collision as far as possible.

Solution:

(i) First consider the braking of car P. The deceleration can be found from:

$$v^2 = u^2 + 2as$$

and with u = 8 m s^{-1}, v = 2 m s^{-1} and s = 10 m, we get

$$a = \frac{v^2 - u^2}{2s} = \frac{4 - 64}{20} = -3\ m\,s^{-2}$$

The braking force required to produce this deceleration is

$$F_{brake} = ma = 1200 \times 3 = 3600J$$

The loss of kinetic energy of car P in the braking process is

$$\Delta(KE) = ½\, mv^2 - ½\, mu^2 = F_{brake}\, s = -1200 \times 3 \times 10 = -36\ kJ$$

(ii) Now suppose we can apply the conservation of linear momentum to the collision. Let car P have velocities v_{Pi} = 2 m s^{-1} and v_{Pf} = 1 m s^{-1} before and after the collision, and likewise the velocities of car Q are v_{Qi} = 0 and v_{Qf}. The conservation of momentum gives

$$m_P v_{Pi} + 0 = m_P v_{Pf} + m_Q v_{Qf}$$

Then

$$v_{Qf} = \frac{m_P (v_{Pi} - v_{Pf})}{m_Q} = \frac{1200(2-1)}{800} = 1.5\ m\,s^{-1}$$

The loss of kinetic energy in the collision is

$$\begin{aligned}\Delta(KE) = KE_f - KE_i &= \tfrac{1}{2} m_P v_{Pf}^2 + \tfrac{1}{2} m_Q v_{Qf}^2 - \tfrac{1}{2} m_P v_{Pi}^2 \\ &= \tfrac{1}{2} 1200 \times (1^2 - 2^2) + \tfrac{1}{2} 800 \times 1.5^2 = -900J\end{aligned}.$$

Notice that the loss of KE in the collision is much less then the loss of KE in the braking process: most of P's KE has been dissipated before the collision.

(iii) Now we work out the impulse from the change in the momentum during the collision: let the impulse be J, then the impulse on P due to collision with Q, J_{PQ} is equal and opposite to J_{QP} the impulse on Q due to collision with P:

$$J_{QP} = \Delta(m_Q v_Q) = m_Q v_{Qf} = 800 \times 1.5 = 1200Ns.$$

The average force on Q in the collision is the impulse divided by the time:

$$F_{collis} = J_{QP} / \Delta t = 1200 / 0.1 = 12000N.$$

The force involved in the collision is therefore much greater than the force involved in braking.

(iv) The change in momentum due to the external forces (braking force) during the collision is only

$$\Delta p_P^{brake} = F_{brake}\Delta t = -3600 \times 0.1 = -360 Ns,$$

while the change in momentum due to the interaction between the cars during the collision is

$$\Delta p_P^{collis} = J_{PQ} = -1200 Ns$$

The effect of the external forces is therefore only 3% of the total effect.

Although this is an artificial problem, it illustrates the essential point, which is that a collision between two bodies results in a large force acting for a short time. In such a short time there is a negligible effect on the momentum due to external forces.

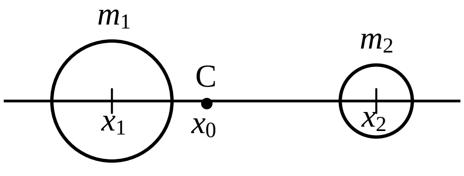
Hence, we can generally use the principle of conservation of momentum, whether there are external forces or not:

$$\boxed{\text{Total linear momentum after a collision} = \text{Total linear momentum before a collision}}$$

We must remember that momentum is a vector quantity, so that both the magnitude and direction of the final momentum must equal the magnitude and direction of the initial momentum.

6.1 Centres of mass

Consider a system consisting of two masses m_1 and m_2. Let the positions of the masses be x_1 and x_2. The centre of mass is at the point C, whose position is x_0, along the line of centres of the masses. To determine the position of the centre of mass we take moments of the masses about C:



$$m_1(x_0 - x_1) = m_2(x_2 - x_0)$$

Rearranging we find

$$x_0(m_1 + m_2) = m_2x_2 + m_1x_1$$

So that

$$x_0 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

From the way we have worked out the position of C, it is clear that if the masses were attached to a light bar and pivoted at C, then the weights of the two masses would balance. We can generalise this treatment to define the position $\underline{r}_0$ of the centre of mass in three dimensions in terms of the position vectors $\underline{r}_1$ and $\underline{r}_2$:

$$\underline{r}_0 = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{m_1 + m_2}$$

We now differentiate with respect to time:

$$\frac{d\underline{r}_0}{dt} = \frac{m_1 \underline{v}_1 + m_2 \underline{v}_2}{m_1 + m_2}$$

The numerator on the right hand side is just the total momentum of the system. It follows that if the total momentum is conserved, then the centre of mass moves with a constant velocity.

6.2 Elastic and inelastic collisions

An *elastic* collision is one in which the total kinetic energy of the bodies after the collision is the same as it was before the collision. An example would be two snooker balls colliding: snooker balls are made of a hard material that does not deform easily, so that energy is not dissipated in a collision.

We saw in the collision between the two cars earlier that there was a considerable loss of kinetic energy in the collision. The energy that disappeared went into deformation of the panel-work, generating heat, sound, and so on. Such a collision, in which the kinetic energy is *not* conserved, is called an *inelastic* collision. The fraction of kinetic energy lost in an inelastic collision depends on the circumstances. However an extreme example, known as a *completely inelastic* collision, is the case where the two bodies stick together and move with a common velocity after the collision.

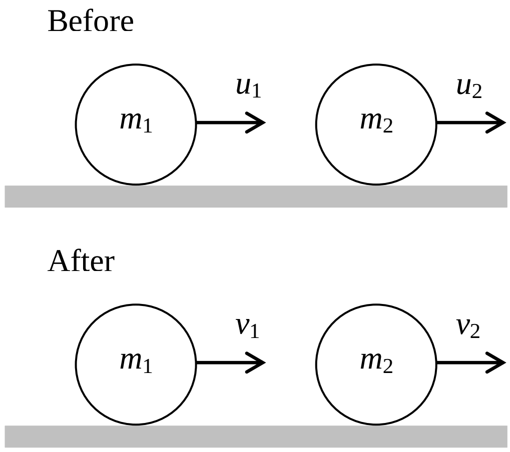
Elastic collision in one dimension

Consider two objects with masses m_1 and m_2 moving along the same straight line.

Let their velocities before and after they collide be u_1, u_2 and v_1, v_2 respectively.

Conservation of momentum yields:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \tag{i}$$



For an elastic collision we can equate the kinetic energies before and after the collision:

$$\tfrac{1}{2}m_1u_1^{\ 2} + \tfrac{1}{2}m_2u_2^{\ 2} = \tfrac{1}{2}m_1v_1^{\ 2} + \tfrac{1}{2}m_2v_2^{\ 2} \tag{ii}$$

Rearranging (i) gives

$$m_1(u_1 - v_1) = -m_2(u_2 - v_2) \tag{iii}$$

From (ii) we get

$$m_1(u_1^{\ 2} - v_1^{\ 2}) = -m_2(u_2^{\ 2} - v_2^{\ 2}) \tag{iv}$$

Using the identity $a^2 - b^2 = (a-b)(a+b)$, equation (iv) becomes:

$$m_1(u_1 - v_1)(u_1 + v_1) = -m_2(u_2 - v_2)(u_2 + v_2).$$

We now divide this equation by equation (iii). This gives:

$$(u_1 + v_1) = (u_2 + v_2)$$

or equivalently

$$(v_1 - v_2) = -(u_1 - u_2) \tag{v}$$

This shows that the relative velocity after an elastic collision is the negative of the relative velocity before the collision. This equation may be combined with the momentum equation, (i) or (iii), to find v_1 and v_2 separately: first, from (v) we write v_2 as:

$$v_2 = (u_1 - u_2) + v_1$$

then substituting into (i) gives

$$v_1 = \frac{2m_2u_2 + u_1(m_1 - m_2)}{m_1 + m_2},$$

whence

$$v_2 = \frac{2m_1u_1 + u_2(m_2 - m_1)}{m_1 + m_2}.$$

Let's look at some special cases:

(a) Equal masses, $m_1 = m_2 = m$

In this case $v_1 = u_2$ and $v_2 = u_1$: the kinetic energies of the masses are interchanged in the collision.

(b) Collision with a stationary ball

In the general case where $m_1 \neq m_2$, but $u_2 = 0$, we find

$$v_1 = \frac{(m_1 - m_2)}{m_1 + m_2}u_1 \qquad \text{and} \qquad v_2 = \frac{2m_1}{m_1 + m_2}u_1$$

In the case of equal masses, e.g. if a snooker ball with velocity u_1 strikes a stationary snooker ball along the line of centres, the first ball stops dead ($v_1 = 0$) and the second ball moves off with velocity $v_2 = u_1$. All the kinetic energy is transferred to the second ball.

(c) Very heavy "target", i.e. $m_2 >> m_1$

In this case we can ignore m_1 compared to m_2. The equations become

$$v_1 = 2u_2 - u_1 \qquad \text{and} \qquad v_2 = u_2$$

This shows that the velocity of the "target" is unaffected by collision with the light projectile, but the projectile has a large change in velocity. Consider a particular case of a gas molecule hitting the wall of its container. If the wall is stationary (u_2 = 0), the molecule bounces straight off with the same speed, but in the opposite direction: $v_1 = -u_1$. Suppose the wall is now moving towards the approaching molecule with velocity $u_2 = -u$. The velocity of recoil of the molecule is now $v_1 = -2u - u_1$, i.e. it picks up *twice* the velocity of the wall, and gains kinetic energy in the process. This is why a gas heats up if it is compressed adiabatically.

(d) Very heavy "projectile", i.e. $m_1 >> m_2$

This is just the inverse of (c), when we can ignore m_2 compared to m_1. The equations are:

$$v_1 = u_1 \quad \text{and} \quad v_2 = 2u_1 - u_2$$

Suppose we fire a cannon ball at a stationary golf ball (u_2 = 0). Then the first equation tells us that the velocity of the cannon ball is unaffected by the collision, but the golf ball picks up *twice* the velocity of the cannon ball! Where does the factor of two come from? Well remember that in an elastic collision the relative velocity after a collision is minus the relative velocity before the collision. In the present case the relative velocity before the collision is $u_1 - u_2 = u_1$. After the collision the relative velocity is $v_1 - v_2 = u_1 - v_2$*, which equals* $-u_1$*, i.e. the negative of the initial relative velocity.*

Question 6b

Two bodies of mass m and 1.5 m move along the same line and undergo an elastic collision. The initial velocities are 1 m s^{-1} and –0.3 m s^{-1}. Calculate the final velocities of the bodies.

Solution: the momentum equation becomes:

$$mv_1 + 1.5mv_2 = m \times 1 + 1.5m \times (-0.3) = m - 0.45m = 0.55m$$

so that

$$v_1 + 1.5v_2 = 0.55 \tag{i}$$

The relative velocity equation gives

$$v_1 - v_2 = -(u_1 - u_2) = -(1-(-0.3)) = -1.3 \qquad \textit{(ii)}$$

Subtracting (ii) from (i) gives

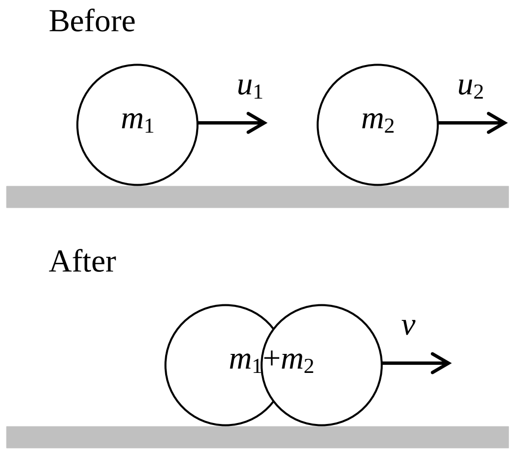
$$2.5v_2 = 1.85\,, \ \ \textit{so} \ \ v_2 = 0.74\ m\,s^{-1},$$

and then

$$v_1 = v_2 - 1.3 = 0.74 - 1.3 = -0.56\ m\,s^{-1}$$

Completely inelastic collisions in one dimension

In a completely inelastic collision the two colliding bodies stick together after the collision and move with a common velocity v as shown in the diagram.



In this case, conservation of momentum gives

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

The final velocity is therefore

$$v = \frac{m_1u_1 + m_2u_2}{(m_1 + m_2)}$$

The kinetic energy lost in the collision is

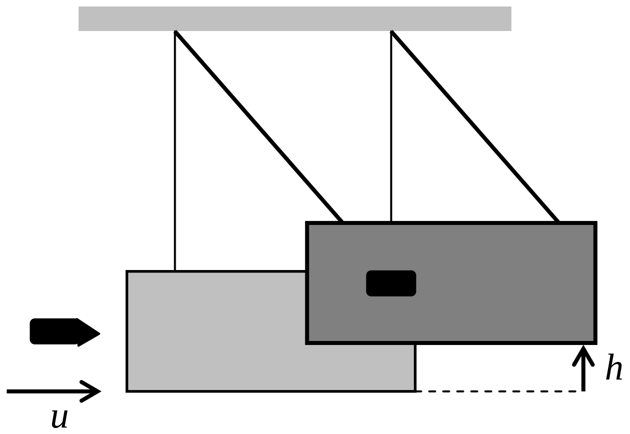
$$\Delta(KE) = \tfrac{1}{2}(m_1 + m_2)v^2 - \tfrac{1}{2}m_1{u_1}^2 - \tfrac{1}{2}m_2{u_2}^2\,.$$

Substituting the value of the common velocity *v* leads to the result:

$$\Delta(KE) = -\frac{m_1m_2}{2(m_1 + m_2)}(u_1 - u_2)^2$$

Question 6c

In the days before high-speed timing devices, the speed of a bullet was measured using a ballistic pendulum, sketched opposite. The bullet of mass m is fired into a wooden block, of mass M, which is supported on two long cords so that it can swing upwards. The maximum height, h, reached by the block is measured. This allows the initial velocity u of the bullet to be estimated.



Suppose that m = 9.5 g and M = 5.4 kg and that the height reached is h = 6.3 cm.

Solution: conservation of momentum gives:

$$mu = (M + m)v \qquad \text{(i)}$$

The kinetic energy of the system (block + bullet) is converted entirely into potential energy when the height is increased to h. It follows that

$$\tfrac{1}{2}(M + m)v^2 = (M + m)gh\,,$$

from which we find

$$v = \sqrt{2gh}$$

Inserting this result into (i) gives

$$u = \frac{M + m}{m}\sqrt{2gh} = \frac{5.4 + 0.0095}{0.0095}\sqrt{2 \times 9.8 \times 0.063} = 632.7\ m\,s^{-1}.$$

How much of the original kinetic energy survives the collision? The initial kinetic energy is $\tfrac{1}{2}mu^2 = \tfrac{1}{2}0.0095 \times 632.7^2 = 1.901 kJ$. *The kinetic energy immediately after the bullet hits the block is equal to the gravitational potential energy acquired by the block and bullet when they have swung up to their ultimate height 6.3 cm. This is*

$$(M + m)gh = 5.4095 \times 9.8 \times 0.063 = 3.3\ J.$$

This is only 0.175% of the initial kinetic energy. The rest goes into deformation of the block, the generation of heat, and so on.

Collisions in two or three dimensions

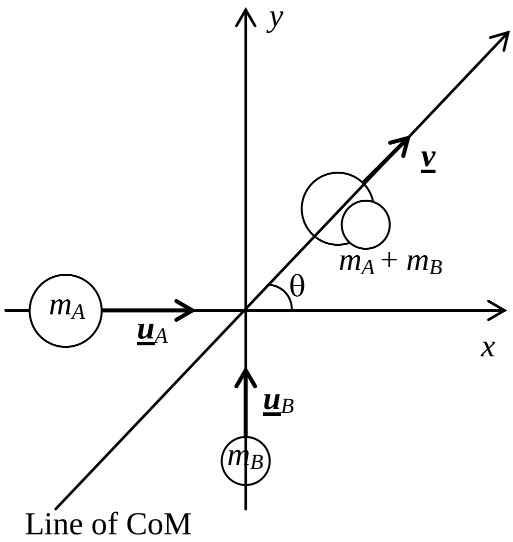
In this case we have to write the conservation of momentum as a vector equation:

$$m_1\underline{u}_1 \;+\; m_2\underline{u}_2 \;=\; m_1\underline{v}_1 \;+\; m_2\underline{v}_2$$

Obviously we can resolve this equation into two or three components, but sometimes it is easier to use the vectors directly.

Question 6d

Two skaters Alf and Bettie collide and embrace, causing a completely inelastic collision. Alf, whose mass is 83 kg, was initially skating due east at 6.2 km/hr. Bettie's mass is 55 kg. She was initially skating due north at 7.8 km/hr. What is the velocity of the pair after the collision?



Solution: conservation of momentum gives:

$$m_A\vec{u}_A + m_B\vec{u}_B = (m_A + m_B)\vec{v} \tag{i}$$

We may first take the x components of the momentum equation:

$$m_A u_A + m_B(0) = (m_A + m_B)v\cos\theta \tag{ii}$$

Taking the y components gives:

$$m_A(0) + m_B u_B = (m_A + m_B) v \sin\theta \qquad \textit{(iii)}$$

We now divide equation (iii) by equation (ii):

$$\frac{m_B u_B}{m_A u_A} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

So that

$$\theta = \tan^{-1}\left(\frac{m_B u_B}{m_A u_A}\right) = \tan^{-1}\left(\frac{55 \times 7.8}{83 \times 6.2}\right) = 39.8^\circ$$

Knowing the angle θ we can now find v from equations (ii) or (iii):

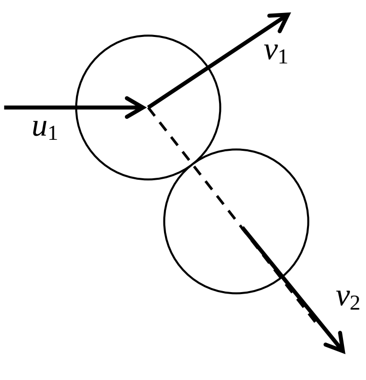
$$v = \frac{m_B u_B}{(m_A + m_B)\sin\theta} = \frac{55 \times 7.8}{(83 + 55) \times 0.64} = 4.86 km/hr$$

So the final velocity of the skaters is 4.86 km/hr at an angle of 39.8° north of east.

Question 6e

A sphere of mass m collides elastically with an identical sphere which is initially at rest. Show that their velocities after collision are at right angles (provided they are both non-zero).

Solution:



Momentum conservation gives $m\underline{u} = m\underline{v}_1 + m\underline{v}_2$, using obvious notation. We can cancel the common factor of m.

Let us take the scalar product of $\underline{u}$ *with itself:*

$$\underline{u} \cdot \underline{u} \;=\; u^2 \;=\; (\underline{v}_1 + v_2) \cdot (\underline{v}_1 + \underline{v}_2) \;=\; {v_1}^2 + {v_2}^2 + 2\underline{v}_1 \cdot \underline{v}_2 \qquad (i)$$

Now we are also told that the collision is elastic, so the kinetic energy is conserved:

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2$$

Cancelling the factor of m/2, we have:

$$u^2 = v_1^2 + v_2^2 \qquad (ii)$$

Subtracting equation (ii) from equation (i) we find

$$\vec{v}_1.\vec{v}_2 = 0$$

It follows that the velocities of the spheres after the collision are at right angles.

Newton's law of restitution

We saw above that for an elastic collision the relative velocity along the line of impact after a collision is equal to minus the relative velocity before the collision:

$$(v_2 - v_1) = -\,(u_2 - u_1)\,.$$

Let us define the relative velocities before and after the collision as

$$u_r = u_2 - u_1 \qquad \text{and} \qquad v_r = v_2 - v_1\,.$$

The condition for an elastic collision is then

$$\boxed{v_r = -u_r}$$

Newton had a very useful insight into what happens in an inelastic collision. He suggested, after making experiments, that if we resolve the velocities along the line of impact, the parallel component of the relative velocity is reversed in sign after the collision, but is reduced by a factor *e* (<1). The component of the relative velocity perpendicular to the line of impact is not affected by the collision. The factor *e* is called the *coefficient of restitution*, and its value depends on the properties of the materials out of which the two objects are made.

Newton's *law of restitution* says, therefore, that for an inelastic collision:

$$\boxed{v_r^{\parallel} = -e u_r^{\parallel}} \qquad \text{and} \qquad \boxed{v_r^{\perp} = u_r^{\perp}}.$$

Here the symbols $^{||}$ and $^{\perp}$ refer to components parallel and perpendicular to the line of impact. This "law" is not exactly and invariably true: but it is a good approximation to what often happens when collisions occur.

Application to inelastic collision in one dimension

Consider two objects with masses m_1 and m_2 moving along the same straight line. Let their velocities before and after they collide be u_1, u_2 and v_1, v_2 respectively. Conservation of momentum yields:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \tag{i}$$

We can introduce relative velocities by writing

$$u_2 = u_r + u_1$$

and

$$v_2 = v_r + v_1$$

Equation (i) can now be rewritten as

$$(m_1 + m_2)u_1 + m_2 u_r = (m_1 + m_2)v_1 + m_2 v_r$$

so that

$$(v_1 - u_1) = -\frac{m_2}{m_1 + m_2}(v_r - u_r) \tag{ii}$$

It also follows from (i) that

$$(v_2 - u_2) = +\frac{m_1}{m_1 + m_2}(v_r - u_r) \tag{iii}$$

The change in the kinetic energy for an inelastic collision is:

$$\Delta(KE) = KE_f - KE_i = \tfrac{1}{2}m_1 v_1{}^2 - \tfrac{1}{2}m_1 u_1{}^2 + \tfrac{1}{2}m_2 v_2{}^2 - \tfrac{1}{2}m_2 u_2{}^2$$

$$= \tfrac{1}{2}m_1(v_1 + u_1)(v_1 - u_1) + \tfrac{1}{2}m_2(v_2 + u_2)(v_2 - u_2)$$

After a bit more manipulation, using equations (ii) and (iii), it is straightforward to show that the loss of kinetic energy in the collision is:

$$\Delta(KE) = \tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_r^2 - u_r^2)\,.$$

If we now use Newton's law : $v_r = -e u_r$, we find

$$\Delta(KE) = -\tfrac{1}{2}\,\frac{m_1 m_2}{m_1 + m_2}\,(1 - e^2)\,u_r^2.$$

This makes an important connection between the coefficient of restitution e and the degree of inelasticity, in terms of how much kinetic energy is lost in the collision. For a perfectly elastic collision, Δ(KE) = 0, and this corresponds to e = 1. For a completely inelastic collision, in which the bodies stick together, their relative velocity is zero, so we deduce that $e = 0$. In this case the loss of kinetic energy is

$$\Delta(KE) = -\tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}u_r^2$$

Bouncing balls

A ball thrown into the air follows a parabolic trajectory, reaching a maximum height

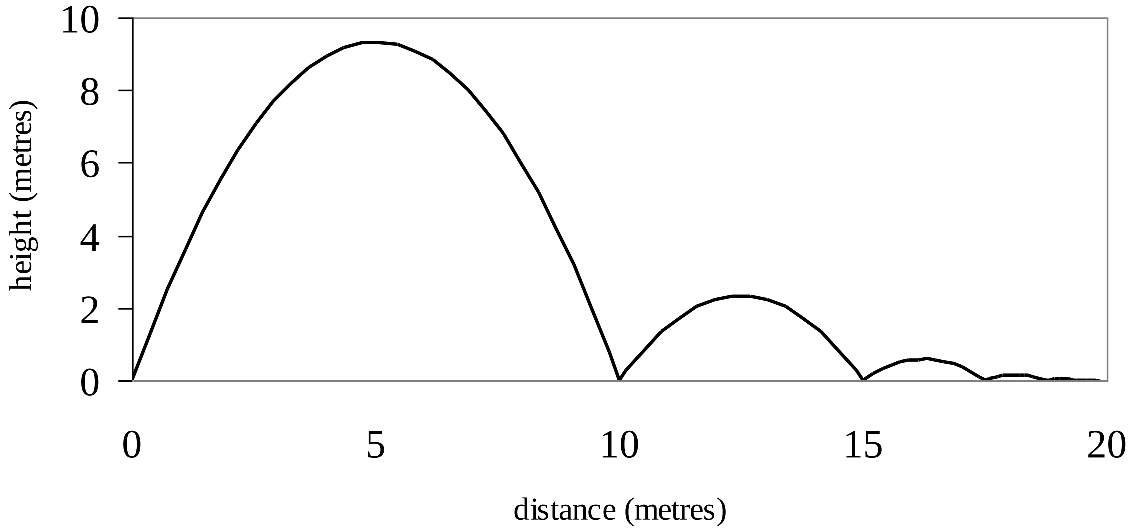
$$z_{\max} = V_z^2 / 2g$$

The horizontal distance travelled before it reaches the ground is

$$x_{\max} = 2\,V_x V_z / g$$

As it reaches the ground its vertical component of velocity is $-V_z$, , which is the negative of its initial value

Suppose the ball now bounces and the coefficient of restitution is *e.* Then the ball's new vertical component of velocity will be + *eV*, but its horizontal component of velocity will be unaffected. After the first bounce the height reached will be smaller by a factor of e^2 but the distance to the next bounce is reduced by a factor of *e*. Each subsequent bounce will be scaled in height and range by additional factors of e^2 and *e* respectively, as shown below:



The trajectory of a ball thrown with a velocity of 14 m s^{-1} at an angle of 75° to the horizontal. The coefficient of restitution between the ball and the ground is 0.5.

Question 6f

A ball thrown vertically reaches a height of 10 metres. If the coefficient of restitution between the ball and the ground is 0.9, how long will it take the ball to come to rest?

Solution: the time to the first bounce is

$$t_1 = 2V_z / g$$

We can find the velocity from the height:

$$z_{\max} = V_z^2 / 2g$$

It follows that

$$t_1 = 2\sqrt{2z_{\max}/g} = 2.857\ s$$

Then the time to the second bounce is et_1, and so on. The total time is

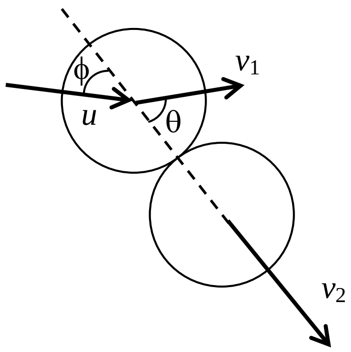
$$t_{tot} = t_1 + et_1 + e^2 t_1 + e^3 t_1 + \cdots = t_1(1 + e + e^2 + e^3 + \cdots)$$

The series in the brackets is just an infinite geometrical progression whose sum is $\frac{1}{1-e}$. It follows that the time for the ball to come to rest is:

$$t_{tot} = \frac{t_1}{1-e} = \frac{2.857}{1-0.9} = 28.57\ s$$

Question 6g

The cue ball in a game of billiards is travelling with a velocity *u* when it strikes a red ball at an angle ϕ to the line of impact. If the coefficient of restitution between the balls is *e*, find the angle at which the cue ball travels after the collision.



Solution: conservation of momentum along the line of impact gives:

$$mu\cos\phi = mv_1\cos\theta + mv_2$$

Cancelling the mass on both sides gives

$$v_1\cos\theta + v_2 = u\cos\phi \qquad \textit{(i)}$$

From Newton's law of restitution the component of relative velocity perpendicular to the line of impact remains unchanged, so that

$$v_1\sin\theta = u\sin\phi \qquad \textit{(ii)}$$

Parallel to the line of impact Newton's law gives:

$$v_2 - v_1\cos\theta = -e(0 - u\cos\phi) = eu\cos\phi \qquad \textit{(iii)}$$

Adding (i) and (iii) leads to:

$$v_2 = \tfrac{1}{2}(1+e)u\cos\phi$$

Subtracting (iii) from (i) gives:

$$v_1 \cos\theta = \tfrac{1}{2}(1-e)u\cos\phi\,.$$

We now divide (ii) by (iii) to give:

$$\frac{v_1 \sin\theta}{v_1 \cos\theta} = \frac{u\sin\phi}{\tfrac{1}{2}(1-e)u\cos\phi}$$

and it follows that

$$\tan\theta = \frac{2\tan\phi}{(1-e)}$$

Notice that for an elastic collision, when e =1, this always gives $\theta = \pi/2$.