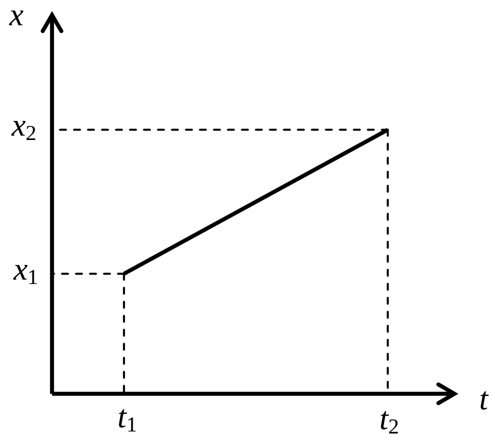
Section 4: Kinematics and Newton's Laws of Motion

Newton's laws tell us how the motion of an object is related to the forces acting on the object. However we start by considering the motion itself and how the distance travelled, speed and time are interrelated.

4.1 Motion along a straight line

Constant velocity

Suppose an object moves at a constant velocity along the x axis in the positive x direction.



The position-time graph is shown opposite. At successive times t_1 and t_2 the positions are x_1 and x_2. The velocity v is then

$$v = \frac{x_2 - x_1}{t_2 - t_1} \quad (\mathrm{m\ s^{-1}})$$

It follows that if the thing has a position x_0 when t=0, its position at a later time t is

$$x = x_0 + vt$$

If we define the displacement s as the distance travelled from the origin,

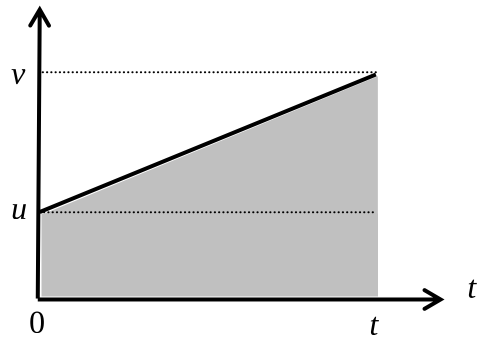
$$s = x - x_0$$

and

$$s = vt$$

The diagram opposite is a graph velocity against time. Here, the displacement s is equal to the area (shaded) under the graph.

Constant Acceleration



Suppose an object initially moving with a velocity u, is subject to a uniform acceleration a.

On a velocity-time graph (see opposite) the velocity is seen to increase linearly with time, the acceleration being given by

$$a = \frac{v_2 - v_1}{t_2 - t_1}$$

where v_1 and v_2 are the velocities at points in time t_1 and t_2. It follows that we may write the velocity v at a time t as

$$v = u + at \tag{i}$$

To find the displacement s, in other words the distance travelled, we calculate the area under the v-t graph. This is equal to the area of the lower rectangle ut plus the area of the upper triangle

$$\tfrac{1}{2}(v-u)t$$

so that

$$s = ut + \tfrac{1}{2}(v-u)t = \tfrac{1}{2}(u+v)t \tag{ii}$$

This makes sense, since the average velocity is

$$\tfrac{1}{2}(u+v)$$

and

distance = average velocity x time

We now use (i) above to substitute for v in (ii) above, giving:

$$s = \tfrac{1}{2}(u+v)t = \tfrac{1}{2}ut + \tfrac{1}{2}(u+at)t = ut + \tfrac{1}{2}at^2 \tag{iii}$$

It is often very useful to have a relation between the velocities and the displacement without involving the time. Equation (i) above can be written in the form

$$t = (v-u)/a$$

and that can be used to substitute for t in (iii). This gives

$$s = u\frac{(v-u)}{a} + \frac{1}{2}a\frac{(v-u)^2}{a^2} = \frac{(v-u)}{2a}(2u+v-u) = \frac{(v^2-u^2)}{2a},$$

leading to

$$v^2 = u^2 + 2as \qquad \text{(iv)}$$

These four equations, (i) to (iv), are very commonly used to solve problems involving linear motion under constant acceleration. They are summarised below:

Equations for linear motion under constant acceleration

These relate the quantities *s, u, v, a, t,* where:

s	is the displacement (= distance travelled, (x-x_0))
u	is the initial velocity
v	is the final velocity
a	is the acceleration
t	is the elapsed time.

(i) $v = u + at$

(ii) $s = \frac{1}{2}(u+v)t$

(iii) $s = ut + \frac{1}{2}at^2$

(iv) $v^2 = u^2 + 2as$

Question 4a

Having spotted a police car, a driver brakes from 100 km/h to 80 km/h over a distance of 88 metres. (a) What is the acceleration, and (b) how long does it take?

A note on terminology: the units km/h (kilometres per hour) are not standard SI units, and the "slash" notation (for division) is something that would normally be deprecated in favour of the inverse power (km h^{-1}). Furthermore, be aware that "hour" is sometimes abbreviated to hr, not h. However, "km/h" is common usage which is unlikely to be misunderstood.

Solution: first express the velocities in metres per second:

$$100 \text{ km/h} = 100/3.6 = 27.77 \text{ m s}^{-1}$$

and

$$80 \text{ km/h} = 80/3.6 = 22.22 \text{ m s}^{-1}.$$

(a) *Using eq. (iv) above, the acceleration is*

$$a = \frac{v^2 - u^2}{2s} = \frac{22.22^2 - 27.77^2}{2 \times 88} = -1.578 \quad \text{m s}^{-2}$$

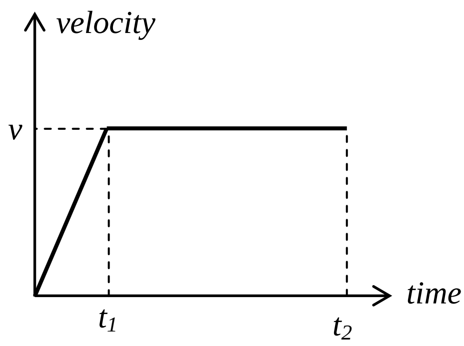
(b) *Now that we know the acceleration, we can use eq. (i) to find the time taken to brake:*

$$t = \frac{v - u}{a} = \frac{22.22 - 27.77}{-1.578} = 3.52 \ s$$

Question 4b

A train accelerates uniformly from rest. It reaches 60 km hr^{-1} in 6 minutes, after which the speed is kept constant. Calculate the total time taken to travel 6 km.

Solution: the velocity-time graph looks like this:



The final velocity of the train must first be expressed in standard units:

$$60 \text{ km hr}^{-1} = (60/3.6) = 16.67 \text{ m s}^{-1}$$

Let the time taken to reach the final velocity v be t_1, and the time to travel the total distance, 6 km, be t_2. Then, the total distance travelled is equal to the area under the v-t graph:

$$s = \tfrac{1}{2} v t_1 + v(t_2 - t_1)$$

But t_1 = 6 minutes = 360 s. So, when s = 6000 metres, we have

$$s = 6000 = \tfrac{1}{2} \times 16.67 \times 360 + 16.67 \times (t_2 - 360)$$

leading to the answer

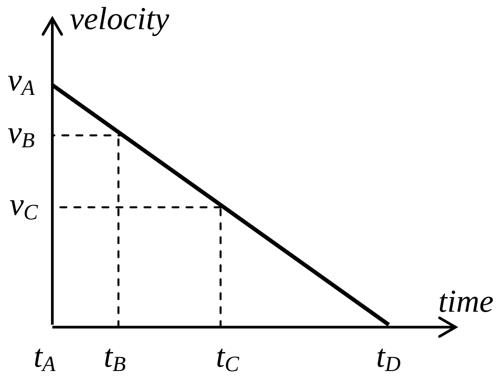
$$t_2 = \left[\frac{6000}{16.67} - 180\right] + 360 = 540\,s$$

or 9 minutes.

Question 4c

A train is slowing down at a steady rate. It is timed between posts A, B and C, spaced 2 km apart: it takes 100 seconds to travel between A and B and 150 seconds to travel between B and C. Find the deceleration of the train and calculate the distance it will travel beyond C before it stops.

Solution:



(a) From the v-t diagram,

$$t_B - t_A = 100 \text{ seconds}$$

and

$$t_C - t_B = 150 \text{ seconds.}$$

The distance travelled from A to B is 2000 metres, so the area under the v-t diagram is

$$s = 2000 = \tfrac{1}{2}\,(v_A + v_A)(t_B - t_A)$$

It follows that

$$v_a + v_B = 2 \ x \ \frac{2000}{100} = 40 \ m \, s^{-1}$$

(i)

The distance between B and C is also 2000 metres, and

$$s = 2000 = ½ \, (v_A + v_C)(t_C - t_B)$$

leading to

$$v_B + v_C = 2 \times 2000/150 = 26.67 \text{ m s}^{-1}$$

(ii)

Subtracting the two equations, (ii) – (i),

$$v_C - v_A = 26.67 - 40.0 = -13.33 \text{ m s}^{-1}$$

The deceleration of the train is therefore

$$a = \frac{v_C - v_A}{t_C - t_A} = \frac{-13.33}{250} = -0.0533 \text{m s}^{-2}.$$

(b) *Now, the distance travelled from A to B is 2000m, so if we use the equation*

$$s = v_A t + \tfrac{1}{2} a t^2$$

we have

$$2000 = v_A \times 100 - \tfrac{1}{2} \times 0.05333 \times 100^2$$

which gives

$$v_A = 22.67 \text{m s}^{-1}$$

Finally, we find the total distance travelled from A to the point D where the train stops from the equation $v^2 = u^2 + 2as$, with $v = 0$ and $u = v_A$. We find

$$s = \frac{v^2 - u^2}{2a} = \frac{-v_A^2}{-2 \times 0.0533} = \frac{22.67^2}{2 \times 0.0533} = 4816.7m$$

So the distance travelled by the train beyond C is

$$4816.67 - 4000 = 816.7m$$

Motion in Free-Fall

A particular example of motion under constant acceleration is a body falling under gravity, where the acceleration is

$$a = -g = -9.8 \text{ ms}^{-2}$$

The minus sign here relates to the fact that the acceleration is *downwards*, and we normally take upward displacements (distances travelled) as positive. Suppose we drop a brick off the top of a building, how far does it fall in successive seconds, and how fast is it travelling? We take the initial velocity $u = 0$, so $s = -\frac{1}{2}gt^2$ and $v = -gt$. The table below lists the values of s and v:

t (seconds)	s (metres)	v(m s^{-1})
0	0	0
1	-4.9	-9.8
2	-19.6	-19.6
3	-44.1	-29.4
4	-78.4	-39.2
5	-122.5	-49.0

Notice that (in magnitude) the velocity increases linearly with time, while the distance travelled increases *quadratically* with time.

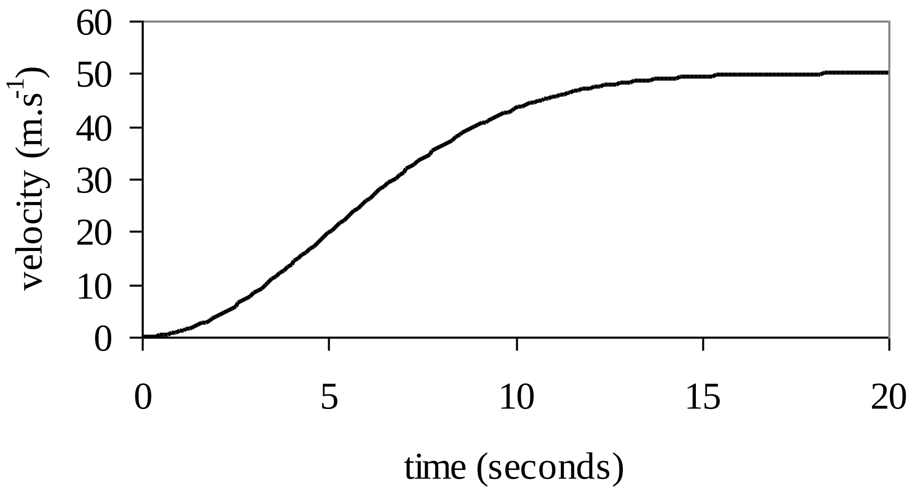
We shall return to motion under gravity in a later section where projectiles are discussed.

Instantaneous Velocity and Acceleration

N.B. This section includes calculus.

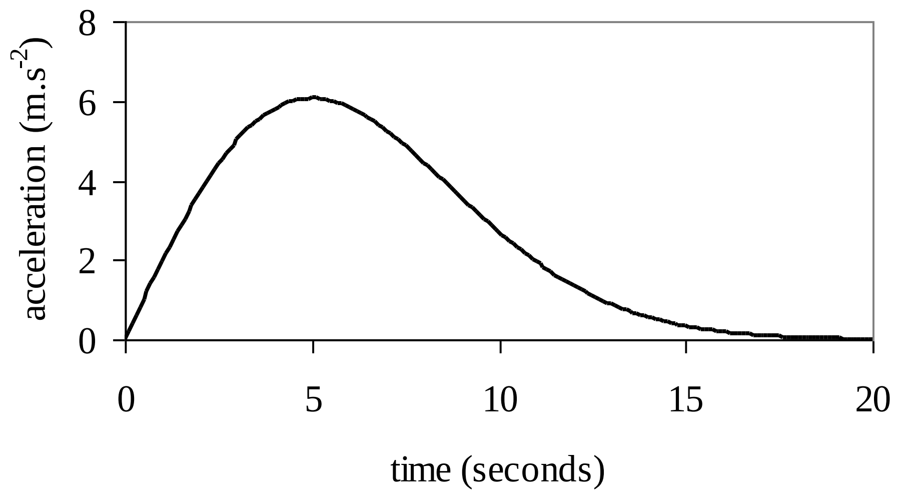
In the case of free fall we can guarantee that the acceleration is constant, as long as we ignore air resistance. In the presence of air resistance the drag force is approximately proportional to the square of the velocity, and eventually becomes practically equal to the downward force due to gravity (the weight of the body): at this point there is no net force acting and the body drops at a constant velocity, the "terminal velocity". In this and many other situations, the acceleration varies with time.

For example, consider the example of a sports car, driven by a speed freak, who drives off down a straight road. The velocity of the car as a function of time might look like this graph:



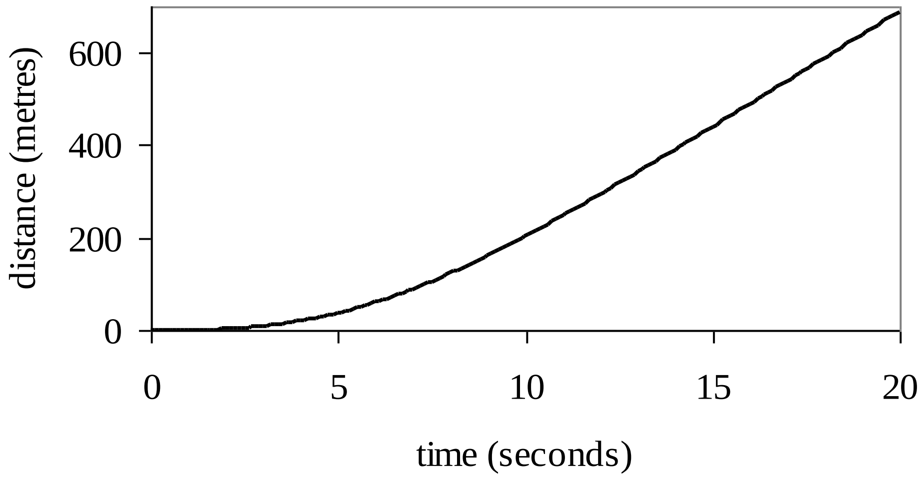
The velocity of a car as a function of time, eventually reaching a maximum speed of 50 m s^{-1}.

It starts from zero, then increases to a maximum value speed of 50 m/s in, say, 20 seconds. Notice that the approach to the maximum speed is gradual, since it is determined by the drag on the car due to the friction of the road and air resistance. The acceleration of the car is the rate of change of the velocity, in other words the *gradient* of the *velocity-time* plot. This is shown in the graph below, where it is evident that the maximum acceleration occurs at about 5 seconds.



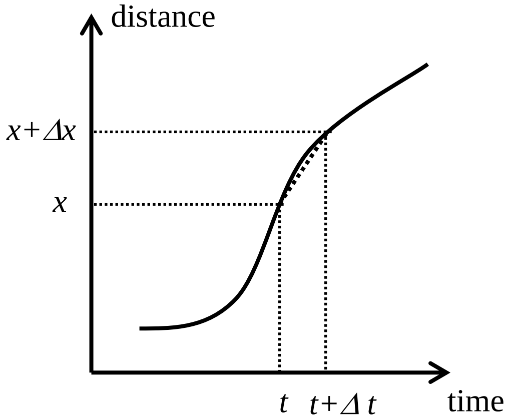
Acceleration as a function of time: the acceleration is equal to the gradient of the velocity-time graph.

As the car approaches its maximum velocity, the acceleration drops to zero (i.e. the car is travelling at a constant speed). The distance travelled is the *area* under the *velocity-time* graph (the first graph above). Notice that after 18 seconds the distance travelled is increasing linearly with time, since the velocity is now practically constant.   A graph of the distance against time is here:



Distance travelled by the car: this is equal to the area under the velocity-time graph.

An important concept is that of "instantaneous velocity". In the graph opposite, the distance travelled is plotted against the elapsed time.



Since the *x-t* curve is non-linear, the velocity varies with time. Suppose we measure *x* at a particular instant *t* and then measure it again a *small* time Δt later. At the time $t+\Delta t$, suppose the distance travelled is $x+\Delta x$.

Then, an estimate of the velocity at time *t* would be

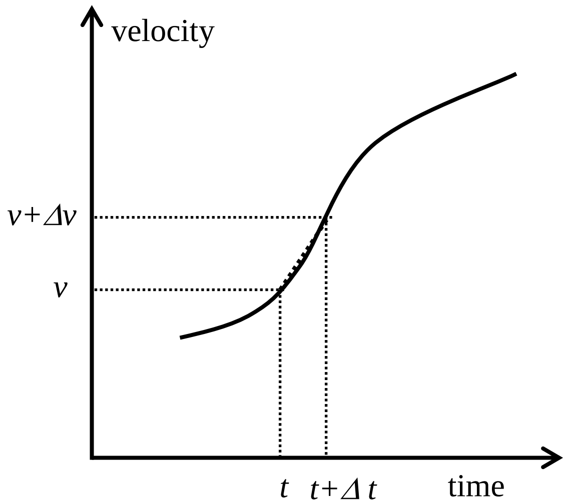
$$v = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

We will get a better approximation to the instantaneous velocity if we make Δt smaller. In fact, if we take the limit $\Delta t \to 0$, the ratio $\Delta x / \Delta t$ becomes the *gradient* or slope of the $x(t)$ graph at the time *t*. We therefore define the instantaneous velocity as

$$v = \left[\frac{\Delta x}{\Delta t}\right]_{Lt(\Delta t \to 0)} = \frac{dx}{dt}$$

The notation $\frac{dx}{dt}$ is called the *derivative* of x with respect to t: it is the standard notation in calculus for the limit as $\Delta t \to 0$ of the ratio $\frac{\Delta x}{\Delta t}$, and therefore defines the gradient of $x(t)$.

We may use the same approach to define the instantaneous value of the acceleration of an object. The sketch opposite is a velocity–time graph.



We may approximate the instantaneous acceleration *a* by measuring the velocity at time *t*, then measuring it again at a slightly later time $t + \Delta t$:

$$a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

We now improve the approximation by making Δt smaller, and we may define the instantaneous acceleration by taking the limit

$\frac{\Delta v}{\Delta t}$ as $\Delta t \to 0$, namely: $a = \left[\frac{\Delta v}{\Delta t}\right]_{Lt(\Delta t \to 0)} = \frac{dv}{dt}$

The symbol $\frac{dv}{dt}$ represents the gradient of the $v(t)$ curve.

4.2 A little calculus

Although calculus is not essential for this course, it can be helpful and it is necessary for higher-level courses in this field. The following is a brief introduction to the key ideas.

Differentiation

For a general function $y(x)$ the derivative is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)$$

and this represents the gradient of the graph of y as a function of x Let us look at some simple examples:

(i) $y = x^2$

$$y(x+\Delta x) = (x+\Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$

It follows that

$$\Delta y = y(x+\Delta x) - y(x) = \Delta x(2x+\Delta x),$$

and so

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = 2x$$

(ii) A general power, $y = ax^n$

To calculate $y(x+\Delta x)$ we use the binomial theorem:

$$y(x+\Delta x) = a(x+\Delta x)^n = a(x^n + nx^{n-1}\Delta x + n(n-1)x^{n-2}(\Delta x)^2 + \cdots)$$

so that

$$\Delta y = y(x+\Delta x) - y(x) = a\Delta x(nx^{n-1} + \Delta x n(n-1)x^{n-2} + \cdots)$$

It follows that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = anx^{n-1}$$

A special case is that the derivative of a constant is zero, i.e. if

$$y = K$$

then

$$\frac{dy}{dx} = 0$$

(iii) Trigonometric functions:

here we quote the results without proof:

$$y = A\sin(ax) \qquad \frac{dy}{dx} = Aa\cos(ax)$$

$$y = B\cos(bx) \qquad \frac{dy}{dx} = -Bb\sin(bx)$$

(iv) The exponential function:

again, without proof:

$$y = C\exp(cx) \qquad \frac{dy}{dx} = Cc\exp(cx)$$

(v) The natural logarithm:

$$y = D\ln(ax) \qquad \frac{dy}{dx} = Da\frac{1}{x}$$

This can be proved easily from the result (iv) for the exponential function, using the fact that in general

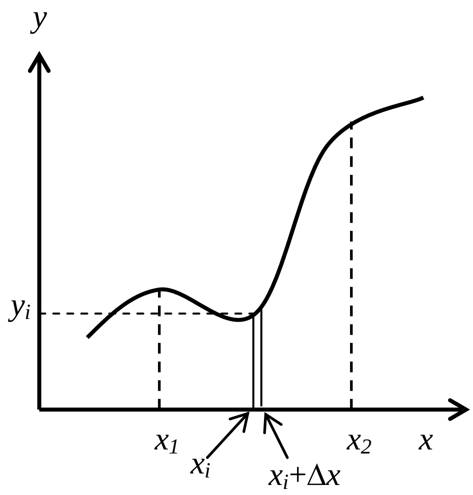
$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$$

Integration

Integration is the complementary mathematical operation to differentiation. We can think of integration as calculating the *area under a graph*.

Consider how to estimate area under the graph of $y(x)$, shown below, between the limits $x = x_1$ and $x = x_2$. We can approximate this area by dividing it up into *N* narrow strips each of width Δx: the area of the strip numbered i would be $y_i \Delta x$, where y_i is the value of $y(x)$ in the centre of strip i. The area is then given approximately by the sum of the areas of all the strips:

$$Area \approx \sum_{i=1}^{N} y_i \Delta x$$



Clearly we get a better approximation if we increase N, the number of strips, and make their widths Δx smaller. So, integral is taken to be the limit where $N \to \infty$ and $\Delta x \to 0$:

$$\int_{x_1}^{x_2} y(x)\,dx = \lim_{N \to \infty, \Delta x \to 0} \left(\sum_{i=1}^{N} y_i \Delta x \right)$$

Some particular examples of integrals are given without proof below:

(i) A general power: $y = ax^n$

$$\int_{x_1}^{x_2} y(x)\,dx = \int_{x_1}^{x_2} ax^n\,dx = a\,\frac{1}{n+1}\,[x^{n+1}]_{x_1}^{x_2} = \frac{a}{n+1}\,[{x_2}^{n+1} - {x_1}^{n+1}]$$

(ii) Trigonometric functions:

$y = A\sin(ax)$: $\quad \int A\sin(ax)dx = -A\frac{1}{a}\cos(ax)$

$y = B\cos(bx)$ $\quad \int B\cos(bx)dx = B\frac{1}{a}\sin(ax)$

(iii) The exponential function:

$y = C\exp(cx)$ $\quad \int C\exp(cx)dx = C\frac{1}{c}\exp(cx)$

The first example, (i) above, which includes the limits of the integration, x_1 and x_2, is called a *definite* integral, while the other examples, where the limits have been omitted, are indefinite integrals.

Applications of Calculus to Linear Motion

Suppose that the position of a particle moving along the x axis is given by

$$x = x_0 + ut + \tfrac{1}{2}at^2,$$

then the velocity v of the particle is the derivative of the position with respect to time:

$$v = \frac{dx}{dt} = u + \tfrac{1}{2}a \times 2t = u + at$$

and the acceleration a is the derivative of the velocity with respect to time, or the second derivative of the position:

$$acceleration = \frac{dv}{dt} = \frac{d^2x}{dt^2} = a\ .$$

Suppose we start with the velocity of the particle,

$$v = u + at$$

then we can find the distance travelled by working out the area under the velocity-time graph: this is just the integral of $v(t)$, with respect to t:

$$x = \int v(t)dt = \int (u + at)dt = ut + \tfrac{1}{2}at^2 + c\ .$$

Notice that when calculating an indefinite integral we should include a constant c in the answer. Any value of the "constant of integration" is compatible with the original expression for the velocity, so we can only determine the position to within the addition of the constant. We need to use extra information to find the value of the constant: for example, if the particle is at x_0 when t=0, then we find $c = x_0$. We then have

$$x - x_0 = ut + \tfrac{1}{2}at^2\ .$$

These results correspond with what was obtained earlier without calculus.

Question 4d

A particle moves along the x axis according to the equation

$$x = 50\,t + 10\,t^{2}.$$

(a) Calculate the average velocity over the first three seconds.
(b) Calculate the instantaneous velocity of the particle at t = 3.0 s.
(c) Calculate the instantaneous acceleration at t = 3.0 s.

Solution:

(a) Firstly, the velocity is

$$v = \frac{dx}{dt} = 50 + 20t\,.$$

This increases linearly with time, so that

$$v(t=0) = 50 m\,s^{-1}$$

and

$$v(t = 3\text{sec}) = 50 + 60 = 110 m\,s^{-1}.$$

The average velocity over the first three seconds is therefore

$$v_{av} = \tfrac{1}{2}(50 + 110) = 80 m\,s^{-1}$$

(b) $v(t = 3\text{sec})\ =\ 110\ m\,s^{-1}$

(c) The acceleration is given by: $a = \dfrac{dv}{dt} = 20\ m\,s^{-2}$ *(constant)*

Question 4e

A particle moves along the x axis with a velocity that changes with time according to the equation

$$v = 10 + 3\,t^{2},$$

where v is in metres per second and t is in seconds. How far has is travelled after a time t= 1s and t= 10s?

Solution: the distance travelled is the area under the v-t graph, or the integral of v with respect to t:

$$x = \int_0^{t_0} v(t)dt = \int_0^{t_0} (10 + 3t^2)dt = [10t + t^3]_0^{t_0} = [10t_0 + {t_0}^3]$$

So for t_0 = 1 second, x=11 m; for t_0 = 10 seconds, x = 1100 m.

Question 4f

A ball bearing drops from rest in a long tube filled with syrup. Its acceleration is initially $-g$, but because of the viscous drag it decreases with time according to:

$$a = -g\exp(-t/\tau)$$

Find its velocity at a time t after it is dropped. What is its terminal velocity?

Solution:

Since the acceleration is the derivative of the velocity:

$$a(t) = \frac{dv}{dt},$$

it follows that velocity is the integral of the acceleration (i.e. the area underneath a graph of the acceleration against time):

$$v(t) = \int a(t)dt = \int (-g)\exp(-t/\tau)dt = (-g)(-\tau)\exp(-t/\tau) + constant$$

To find the constant of integration, suppose that at t=0 the velocity v is zero. Then

$$0 = g\tau + constant$$

so that the constant is equal to $-g\tau$. The result is then

$$v(t) = -g\tau + g\tau\exp(-t/\tau)$$

Rearranging, we find

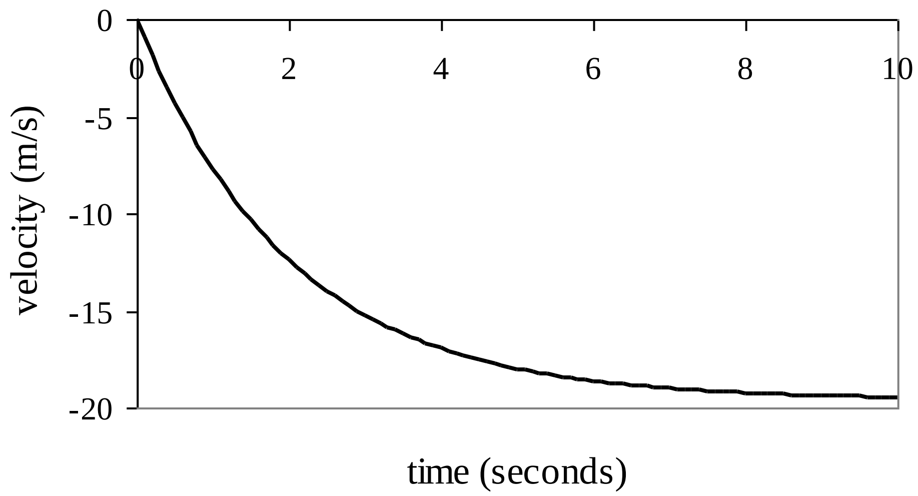
$$v(t) = -g\tau[1 - \exp(-t/\tau)].$$

To find the terminal velocity, we take the limit $t \to \infty$ and find that the velocity tends to the constant value

$$v_\infty = -g\tau$$

The velocity equation may be rewritten using the terminal velocity:

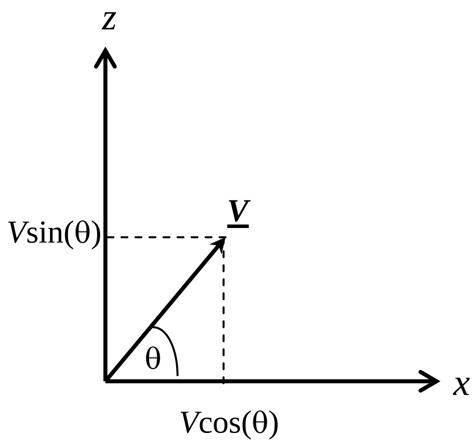
$$v(t) = v_\infty[1 - \exp(-t/\tau)]$$



The velocity of a body dropping in a viscous liquid. The curve has been calculated for $g = 9.8$ m s^{-2}, $\tau = 2$ secs and $v_0 = 0$. The terminal velocity is -19.6 m s^{-1}.

4.3 Projectiles in two dimensions

To investigate the trajectory of a projectile we will apply the equations of linear motion with constant acceleration.



Consider a cannon-ball emerging from the muzzle of a cannon with a velocity $\underline{V}$. The barrel of the cannon is inclined at an angle θ to the horizontal. We take our axes as x horizontally and z vertically, as shown in the sketch. It follows that the horizontal component of the velocity is

$$V_x = V\cos(\theta)$$

and the vertical component of the velocity is

$$V_z = V\sin(\theta)\,.$$

We neglect air resistance. In this example the only force acting on the cannon-ball is that due to gravity, which acts vertically downwards. This means that the acceleration of the cannon-ball is

$$\underline{a} = -\underline{g} = -9.81\underline{\hat{z}} \text{ m s}^{-2}.$$

Here $\underline{\hat{z}}$ is the unit vector in the $\underline{z}$ direction. This means that there is no component of acceleration in the x direction, so the x-component of the velocity remains unaltered. It follows that the displacement of the cannon-ball in the x-direction is given by:

$$x = V_x t = V\cos(\theta)t \qquad \text{(i)}$$

The vertical motion has the uniform acceleration $-g$, so the vertical displacement of the cannon-ball is given by:

$$z = V_z t - \tfrac{1}{2}gt^2 = V\sin(\theta)t - \tfrac{1}{2}gt^2 \qquad \text{(ii)}$$

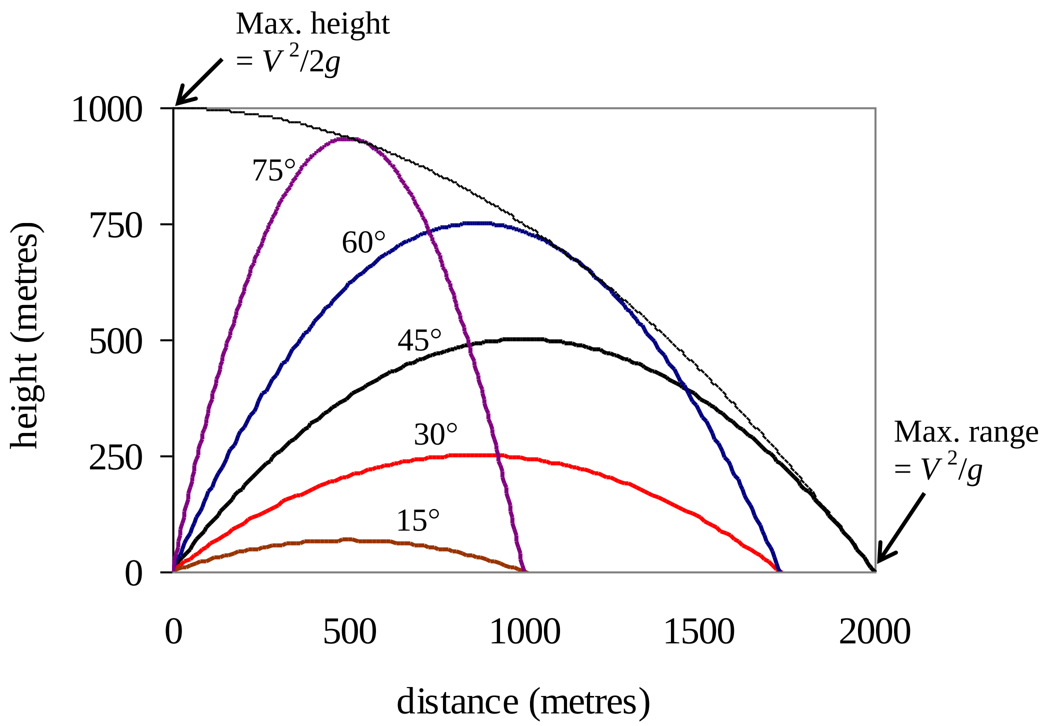
To find the path taken by the cannon-ball we can take

$$t = x / V\cos(\theta)$$

from (i) above and substitute it into (ii):

$$z = x\tan(\theta) - \frac{gx^2}{2V^2\cos^2(\theta)} \tag{iii}$$

This trajectory is a parabola. The graph below shows the trajectories for different angles of projection θ:



Trajectories of a projectile showing parabolic paths for several elevation angles but the same projection speed V=140 m s^{-1}. The dotted curve is the curve of safety, which is the envelope of all points that can be reached for this value of V.

Time of flight of the projectile

How long does the projectile take before it hits the ground? In equation (ii) above we can put z = 0 and find the roots of the equation: one root is t = 0, which is the time of projection, the other root is given by

$$V\sin\theta \quad = \quad \tfrac{1}{2}\,gt$$

from which we see that the time of flight of the projectile is

$$t_{MAX} \quad = \quad \frac{2V\sin\theta}{g}$$

Range of the projectile

To find how far the projectile travels from the origin (along the x-direction) we put the time of flight into the equation (i) above, which describes the x-motion:

$$x_{MAX} \quad = \quad V\cos(\theta)\ .t_{MAX} \quad = \quad \frac{2V^2\ \sin(\theta)\cos(\theta)}{g} \quad = \quad \frac{V^2\sin(2\theta)}{g}$$

Notice we can also write the range in terms of V_x and V_z:

$$x_{\max} = \frac{2V_x V_z}{g}$$

The maximum possible range for a given projection speed occurs when $\sin(2\theta) = 1$, namely when $\theta = \pi/4$ or 45°. In this case,

$$x_{\max} = V^2 / g \ .$$

The maximum height of a projectile

Referring to the trajectories in fig. 4.5, it can be seen that the maximum height occurs exactly half way along the path, namely at a time

$$t = \frac{t_{\max}}{2} = \frac{V\sin(\theta)}{g} .$$

Inserting this value into equation (ii) gives the maximum height:

$$z_{\max} = V\sin(\theta)\frac{V\sin(\theta)}{g} - \tfrac{1}{2}g\left(\frac{V\sin(\theta)}{g}\right)^2$$

This gives

$$z_{\max} = \frac{V^2\sin^2(\theta)}{2g} .$$

The maximum possible height for the projectile is when $\sin(\theta) = 1$, corresponding to $\theta = \pi/2$ or 90°, i.e. when the projectile is launched vertically upwards. Then, the maximum possible height is

$$z_{\max} = \frac{V^2}{2g}.$$

The "curve of safety"

The dotted line in the previous graph is sometimes called the "curve of safety". It is the envelope of the most extreme parts of the trajectory that can be reached for a given projection speed *V*. The equation that describes the curve of safety is:

$$z = \frac{V^2}{2g} - \frac{g}{2V^2} x^2$$

An object *outside* the region encompassed by the curve of safety could not be struck by the projectile.

Question 4g: the formula for the "curve of safety"

A projectile is launched at a speed *V* at a target placed a distance *d* away at a height *h* above a horizontal plane. Show that the projectile can hit the target as long as

$$h \leq \frac{V^2}{2g} - \frac{g}{2V^2} d^2.$$

Solution:

Suppose the angle at which the projectile is launched is θ*, then the horizontal distance travelled in time* t *is*

$$x = V\cos\theta\, t \qquad (i)$$

and the vertical distance is

$$z = V\sin(\theta)t - \tfrac{1}{2}g t^2 \qquad (ii)$$

Substituting t *from (i) :*

$$t = \frac{x}{V\cos\theta}$$

into equation (ii) gives

$$z = x\tan(\theta) - \frac{gx^2}{2V^2}\sec^2\theta$$

(iii)

We now use the trigonometrical identity $\sec^2\theta = 1 + \tan^2\theta$*, so that (iii) becomes a quadratic equation in* $\tan\theta$*. We now put in the extreme values of x and z, namely x = d and z = h. Equation (iii) then becomes*

$$\frac{gd^2}{2V^2}(1 + \tan^2\theta) - d\tan\theta + h = 0$$

Rearranging in powers of $\tan\theta$*:*

$$\frac{gd^2}{2V^2}\tan^2\theta - d\tan\theta + \left[h + \frac{gd^2}{2V^2}\right] = 0 \qquad \textit{(iv)}$$

Now, for a quadratic equation of the form $ax^2 + bx + c = 0$*, the condition that the roots are real is that* $b^2 \geq 4ac$*. Applying this condition to equation (iv) gives:*

$$d^2 \geq 4\frac{gd^2}{2V^2}\left[h + \frac{gd^2}{2V^2}\right].$$

Simplifying, this leads to the required result

$$h \leq \frac{V^2}{2g} - \frac{gd^2}{2V^2}.$$

Question 4h

The baseline of a tennis court is 11.5 m from the net, which is 90 cm high. The server strikes the ball at a height of 2.4 m above the ground with its initial velocity horizontal.

(a) What is the minimum velocity of serve that will allow the ball to clear the net? For the service to be good the ball has to strike the ground, at the most, at 17.5 m from the baseline.

(b) What is the maximum velocity of serve for the ball to be "in", assuming its initial velocity is horizontal? Will this serve clear the net?

Solution:

(a) To clear the net the ball must fall less than 1.5 m in the time interval t taken for the ball to reach the net. Using $s = \frac{1}{2} g t^2$*, this gives*

$$t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2 \times 1.5}{9.8}} = 0.553 s$$

In this time the ball must travel a distance of 11.5 m horizontally, giving a velocity $v = 11.5 / 0.533 = 20.8 \; m s^{-1}$*.*

(b) The time taken for the ball to hit the ground is

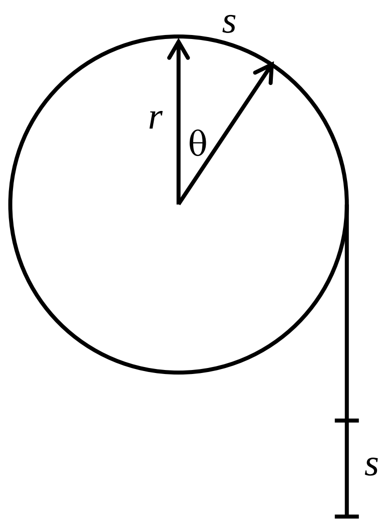
$$t = \sqrt{2 \times 2.4 / 9.8} = 0.7 \; s$$

The maximum velocity is therefore $v = 17.5 / 0.7 = 25 \; m s^{-1}$*. In this case the ball will reach the net in 0.46 seconds and has fallen a distance 1.04 m, so it will clear the net by 46 cm.*

The maximum velocity of 25 m s^{-1} *seems rather slow compared to the serving speeds of ~ 50 m* s^{-1} *usual in top-class tennis. This suggests that the angle of serve should be below the horizontal for higher-velocity serves.*

4.4 Rotational motion

Consider a cylindrical drum of radius r around which is wrapped a cord. Suppose the end of the cord is pulled through a distance s, as in the diagram.



The drum rotates through an angle θ, such that the arc subtended at the circumference is equal to s, and $s = r\theta$.

This follows from the definition of an angle in radians:

$$\theta = \frac{arc\ subtended}{radius\ of\ circle} = \frac{s}{r}\ .$$

Now suppose that the cord is being pulled at a constant rate. In this case the end of the cord would move at a constant velocity v and the drum would rotate at a constant *angular velocity* ω. The angular displacement of the drum would be $\theta = \omega t$. The angular velocity ω is measured in units of radians per second. It is common for rotational speed of a drum or wheel to be reported as, say, N revolutions per minute (rpm). Since one revolution corresponds to 2π radians, it follows that the conversion from rpm to radians per second is:

$$\omega \;=\; \frac{2\pi N}{60} \quad radian\ s^{-1}$$

The relation between linear and angular velocity

Starting from $s = r\theta$ we can derive the relation between linear and angular velocities. The linear velocity v at which the end of the cord moves can be calculated from the linear displacement in a certain time:

$$v = \frac{s_2 - s_1}{t}$$

We can now relate the linear displacement to the angular displacement:

$$v = \frac{s_2 - s_1}{t} = \frac{r\theta_2 - r\theta_1}{t} = r\frac{\theta_2 - \theta_1}{t} = r\omega .$$

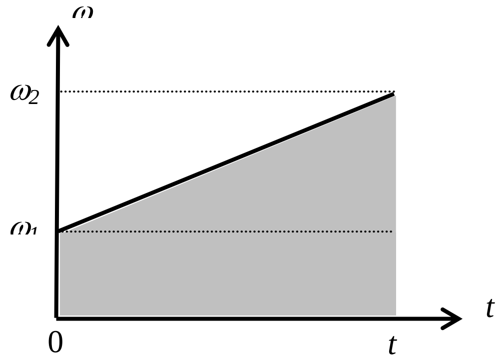
This result also follows directly from $s = r\theta$ by differentiating both sides with respect to time:

$$\frac{ds}{dt} = r\frac{d\theta}{dt},$$

giving

$$\boxed{v = r\omega}.$$

Angular acceleration



Suppose that the cord is pulled with a uniform linear acceleration *a*, so that the end of the cord has a velocity

$$v = u + at$$

where u is its initial velocity, and its linear displacement is

$$s = ut + \tfrac{1}{2}at^2$$

It follows that the angular velocity will increase linearly with time, from a value $\omega_1 = \frac{u}{r}$ to a value $\omega_2 = \frac{v}{r}$.

We may therefore define the *angular acceleration* α as

$$angular\ acceleration = \alpha = \frac{\omega_2 - \omega_1}{t},$$

so that

$$\boxed{\omega_2 = \omega_1 + \alpha t} \qquad \text{(i)}$$

It follows that

$$\alpha = \frac{1}{r}\frac{v-u}{t} = \frac{a}{r},$$

i.e. the relation between the linear acceleration a and the angular acceleration α is

$$\boxed{a = \alpha r}$$

This also follows directly by differentiating both sides of the relation $v = r\omega$ with respect to time:

$$\frac{dv}{dt} = r\frac{d\omega}{dt}.$$

Since $\omega = \dfrac{d\theta}{dt}$ it follows that the angular displacement is the area under a plot of ω against time t.

For a constant angular acceleration it follows that the angular displacement is given by the area under a graph (such as the one above) of ω as a function of time. We get

$$\theta = \omega_1 t + ½(\omega_2 - \omega_1)t$$

so that

$$\boxed{\theta = ½(\omega_2 + \omega_1)t}$$

(ii)

We may also write, from the definition of angular acceleration:

$$\omega_2 - \omega_1 = \alpha t$$

Substituting this into the result for the area, we have

$$\boxed{\theta = \omega_1 t + \tfrac{1}{2}\alpha t^2}$$

(iii)

Finally from (ii) we relate the time to the displacement: $t = \dfrac{2\theta}{\omega_2 + \omega_1}$ and eliminate t in the definition of the angular acceleration so as to get

$$\omega_2 - \omega_1 = \alpha t = \alpha\frac{2\theta}{\omega_2 - \omega_1}.$$

which can be rearranged to give

$$\boxed{\omega_2^2 = \omega_1^2 + 2\alpha\theta}$$

(iv)

Each of these equations is a counterpart to one of the equations for linear motion. Both sets of equations are summarised below:

Summary of equations for linear and rotational motion

	Linear Motion	**Rotational Motion**
(i)	$v = u + at$	$\omega_2 = \omega_1 + \alpha t$
(ii)	$s = \frac{1}{2}(u + v)t$	$\theta = ½\,(\omega_1 + \omega_2)\,t$
(iii)	$s = ut + \frac{1}{2}at^2$	$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
(iv)	$v^2 = u^2 + 2as$	$\omega_2^2 = \omega_1^2 + 2\alpha\theta$

Question 4i

The speed of an electric motor increases linearly from zero to 500 rpm in 10 s. Find the angular acceleration and the total number of revolutions made in the 10 s.

Solution:

$\omega_1 = 0$, $\omega_2 = 500 \times 2\pi/60$ radians s^{-1}

so that

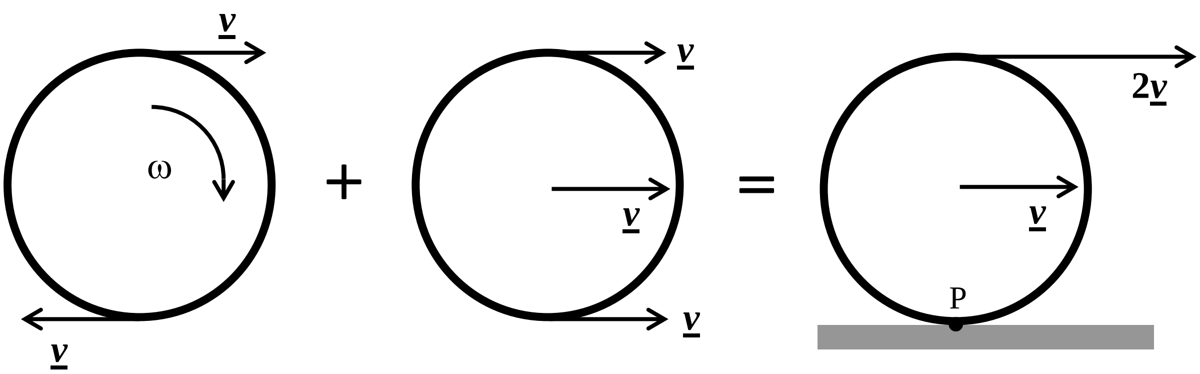
$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{500 \times 2\pi}{60 \times 10} = 5.236 \text{ radians s}^{-1}$$

$$\theta = ½\,(\omega_1 + \omega_2)\,t = \frac{500 \times 2\pi}{2 \times 60} \times 10 = 262\,radians$$

$$= 41.7\,revs$$

Rolling motion

When a wheel rolls steadily along a surface at a constant rate, the centre of the wheel travels with a fixed linear velocity v but a point on the rim has a more complex motion, being a combination of rotation about the centre and translation, as shown here:

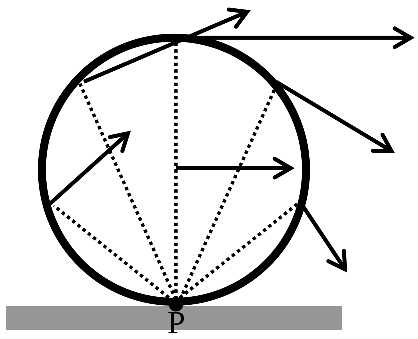


(a) Pure rotation (b) Pure translation (c) Rolling motion

Rolling motion as a combination of pure rotational motion and pure translational motion. Note that in (c) the point P at which the wheel is in contact with the ground is stationary. This is a requirement for rolling.

The condition for rolling is that there should be no relative motion between the point of the wheel in contact with the surface and the surface itself: if this condition is not met the wheel will slide or skid across the surface. It follows that the point P in (c) above is instantaneously stationary. Its forward translational velocity *v,* due to the forward motion of the whole wheel (as in (b)) must be cancelled exactly by the backward velocity due to the rotation of the wheel (as in (a)). This means that the velocity of the wheel's rim relative to its centre must be exactly equal to the linear forward velocity v of the centre of the wheel. This tells us that the relation between the forward velocity of the wheel and its angular velocity of rotation is $v = \omega r$.

If we combine the instantaneous velocities due to rotation with the forward translational velocity for each point on the wheel, the motion appears to be equivalent to an instantaneous rotation about the point of the wheel in contact with the surface, as shown in the following diagram.



The combination of rotation and translation is equivalent to an instantaneous rotation about the point P, at which the wheel is in contact with the road.

4.5 Newton’s Laws of Motion

Newton’s laws of motion have already been used implicitly in the definition of force and in the context of static equilibrium.

Newton’s First Law: *a body will remain at rest or continue to move at a constant velocity unless acted on by an external force.*

This is equivalent to saying that

if no net force acts on a body, then the body’s velocity cannot change: in other words, it cannot accelerate.

It follows that in static equilibrium, where a system is either stationary or is moving with a constant velocity, the net force is zero.

Newton’s Second Law: *the rate of change of the momentum of a body is equal to the net force acting on the body and it takes place in the direction of the net force.*

The standard symbol for momentum is $\underline{p}$ (= $m\underline{v}$). Suppose the initial momentum of an object of mass m is $\underline{p}_1$ (= $m\underline{v}_1$) and it is then subjected to a constant force $\underline{F}$ for a time t. Then, its final momentum will be $\underline{p}_2$ (= $m\underline{v}_2$) and

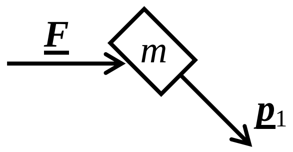
$$\underline{F} = \frac{\underline{p}_2 = \underline{p}_1}{t} = \frac{m\left(\underline{v}_2 - \underline{v}_1\right)}{t} = m\underline{a}$$

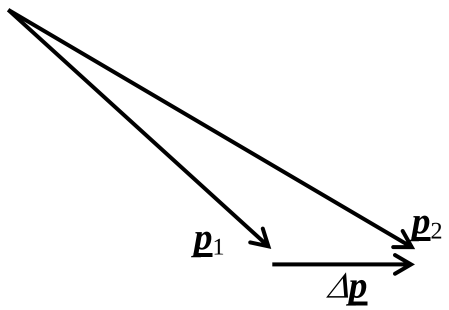
where $\underline{a}$ is the acceleration of the body. Note that this is a vector equation: the direction of the acceleration is parallel to the direction of the applied force. This result gives an alternative statement of Newton's second law:

When a net force F acts on a body it produces an acceleration $\underline{a}$ *equal to the ratio of the force to the mass of the body; the direction of the acceleration is parallel to the direction of the net force:*

$$\underline{F} = m\underline{a}$$

To emphasise the vector nature of Newton's second law, consider the case where the applied force is at an angle to the initial momentum of the body:





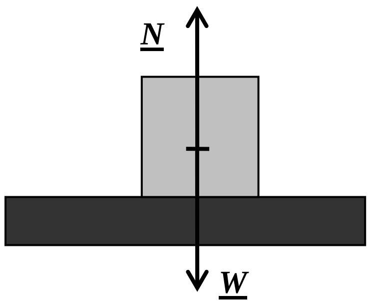
The vector diagram on the right shows that $\Delta\underline{p}$, the change in momentum, is parallel to the direction of application of the force $\underline{F}$.

We have already seen that Newton's second law defines the unit of force, the newton, as *the force which acting on a mass of 1 kilogram, produces an acceleration of 1 m s*$^{-2}$.

Newton's Third Law: *When two bodies interact, the forces on the bodies from each other are always equal in magnitude and opposite in direction.*

We have used this law implicitly when discussing, for example, the normal reaction force between a block and the table on which the block rests.

The block acts on the table with a force equal to its weight $\underline{W}$. By Newton's third law the table acts back on the block with the reaction force $\underline{N}$, equal and opposite to the weight:



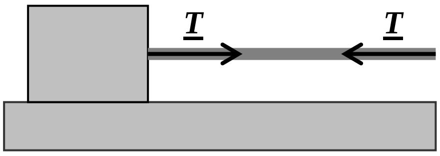
$$\underline{N} = -\underline{W}$$

It follows that the net force vanishes:

$$\underline{N} + \underline{W} = 0$$

so that the block is in static equilibrium.

Similarly, if we pull on a rope attached to a large mass, the tension in the rope transmits a force $\underline{T}$ to the mass. This is equal and opposite to the force we exert on the end of the rope.



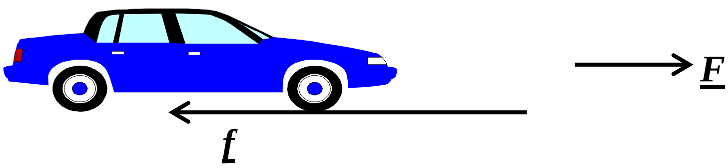
Inertial and Gravitational Mass

We routinely find out the mass of an object by measuring its weight, i.e. by measuring the magnitude of the gravitational force mg acting on it. This can be called the "gravitational mass" of the object. But we could also measure the mass of an object without reference to gravitation by measuring the acceleration given to it by some other kind of force (e.g. by a motor pulling it horizontally). This can be called the "inertial mass". In Newtonian mechanics there is no *a priori* reason why the gravitational mass and the inertial mass should be the same, but in practice they appear to be identical. A Hungarian physicist, Eötvös, carried out very precise measurements that showed no difference between inertial and gravitational mass to one part in 10^7. Einstein concluded that this cannot be a coincidence, and so he took it that there must be a fundamental equivalence between inertial and gravitational masses and he used this principle as one of the foundations of his theory of general relativity.

Applications of Newton's Laws of Motion

Question 4j

A car of mass 1 tonne accelerates from rest to 100 m s^{-1} in 20 s. If the resistance to motion is 100 N, determine the traction force required.



Solution:

Let the traction force be F and the resistive force be f. The net force acting on the car is then F –f. The acceleration of the car is $a=(v-u)/t=100/20=5\ m\,s^{-2}$*. The equation of motion for the car is* $\vec{F}-\vec{f}=m\vec{a}$*, so that* $|\vec{F}-\vec{f}\,|=ma=1000\times 5\ =\ 5000\ \ N$*.*

Since $f=100N$*, it follows that the traction force F = 5100 N.*

Question 4k

Coal wagons are hauled along a level track by a winch. The total mass of the wagons is 20 tonnes. If the resistance to motion is 1500 N per tonne, calculate the tension in the winch cable when

(a) the wagons are accelerating at 5 m s^{-2}, and

(b) the wagons are decelerating at 1.25 m s^{-2}.

Solution:

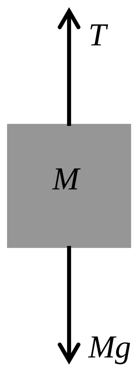
Let T be the tension in the cable and f be the frictional force, whose magnitude is 1500 $\times$ *20 = 30000 N. The net force on the wagons will then be T–f and the equation of motion is* $|\vec{T}-\vec{f}\,|=Ma$*. It follows that* $T=Ma+f$*.*

(a) $T=Ma+f=20\times 10^{3}(kg)\times 5(m\,s^{-2})+30\times 10^{3}=130\ kN$*.*

(b) $T=20\times 10^{3}(kg)\times(-1.25)(m\,s^{-2})+30\times 10^{3}=(30-25)\ kN=5\ kN$

Question 4l

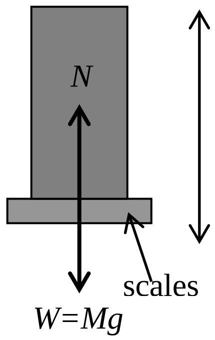
A mass of 400 kg is raised vertically from rest by a winch at a constant acceleration. It attains a velocity of 6 m s^{-1} after 4 seconds. What is the tension in the winch cable?



Solution: the net upward force on the mass is $T-Mg$*, so the equation of motion is* $T-Mg=Ma$*. The acceleration of the mass is* $a=(v-u)/t=6/4=1.5\ m\,s^{-2}$*, so the tension in the cable is* $T=M(a+g)=400\times(1.5+9.8)=4520N$*.*

Question 4m

A passenger in a lift stands on bathroom scales. The passenger's mass is 72.2 kg. What is the reading on the scales (calibrated in newtons) if:



(a) the lift is stationary,

(b) the lift moves upwards at a constant speed of 1 m s^{-1},

(c) the lift accelerates upwards at 3.2 m s^{-1},

(d) the lift accelerates downwards at 3.2 m s^{-1}?

Solution:

(a) *When the lift is stationary the passenger exerts a force on the scales equal to his weight Mg. The normal reaction force of the scales on the passenger is equal and opposite to the weight, so its magnitude is N = Mg. This normal reaction force is created in the scales by compression of a spring: it follows that the reading on the scale is equal to N. The reading is therefore 72.2 $\times$ 9.8 = 707.6 N.*

(b) *When the lift is moving at constant speed, there is no additional acceleration on the passenger or the scales, so we still have N = Mg. The reading on the scales is still 707.6 N.*

(c) *For the passenger to be accelerated upwards, she must be subjected to a net upward force. This can only come from the normal reaction force. The net upward force acting on the passenger is $\underline{N} - M\underline{g}$. This produces an upward acceleration $\underline{a}$. The equation of motion for the passenger is then*

$$\vec{N} - M\vec{g} = M\vec{a}$$

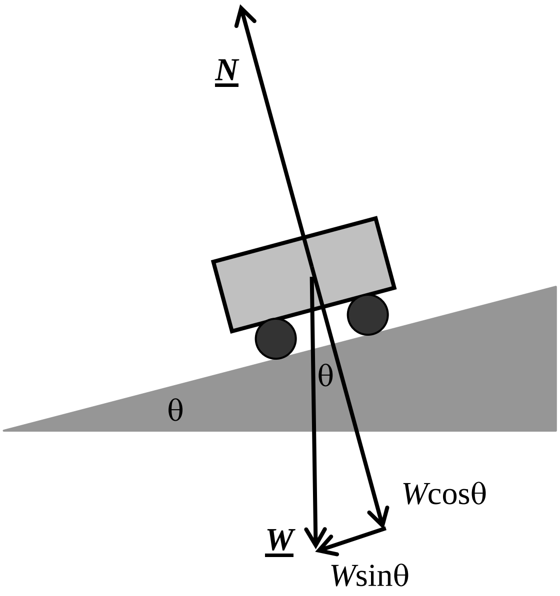
so the magnitude of the reaction force, and therefore the reading on the scale, is

$$N = M(g + a) = 72.2 \times (9.8 + 3.2) = 938.6N .$$

(d) *When the acceleration is downwards, a = -3.2 m s^{-2}, the reaction force and hence the scale reading is:*

$$N = M(g + a) = 72.2 \times (9.8 - 3.2) = 476.5N .$$

Question 4n



A truck of mass 1 tonne is held stationary on a plane inclined at a gradient 1:30 to the horizontal. If the truck is released, calculate its acceleration and the distance it travels in 20 seconds. Ignore friction.

Solution:

The only forces acting are the weight $\underline{W} = M\underline{g}$ *acting vertically downwards and the normal reaction force* $\underline{N}$*, which is perpendicular to the plane.* $\underline{N}$ *is balanced by the component of the weight normal to the plane:*

$$N = W\cos\theta \ .$$

In the absence of friction the net force acting down the slope is the component of the weight, so the equation of motion of the truck is

$$W\sin\theta = Mg\sin\theta = Ma\,.$$

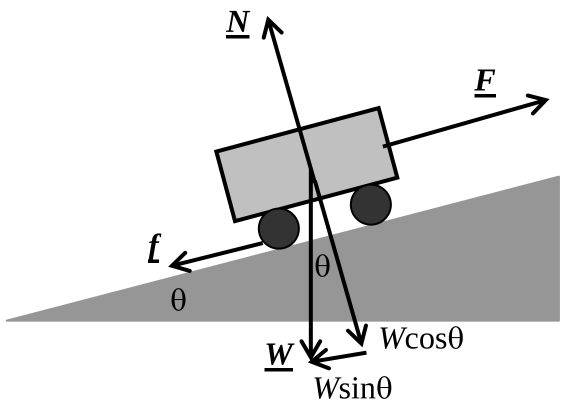
The acceleration of the truck is therefore

$$a = g\sin\theta = 9.8\times(1/30) = 0.3267ms^{-2}$$

Starting from rest the distance travelled is

$$s = \tfrac{1}{2}at^2 = 0.5\times 0.3267\times(20)^2 = 65.33m\,.$$

Question 4o



A truck of mass 1.6 tonnes accelerates up an incline with a gradient of 1:40 under a traction force of 3 kN. If the resistance to motion is 200 N per tonne calculate the acceleration of the truck.

Solution: let F be the traction force and f the frictional force. The other forces in the problem are the weight of the truck W and the normal reaction force N as illustrated.

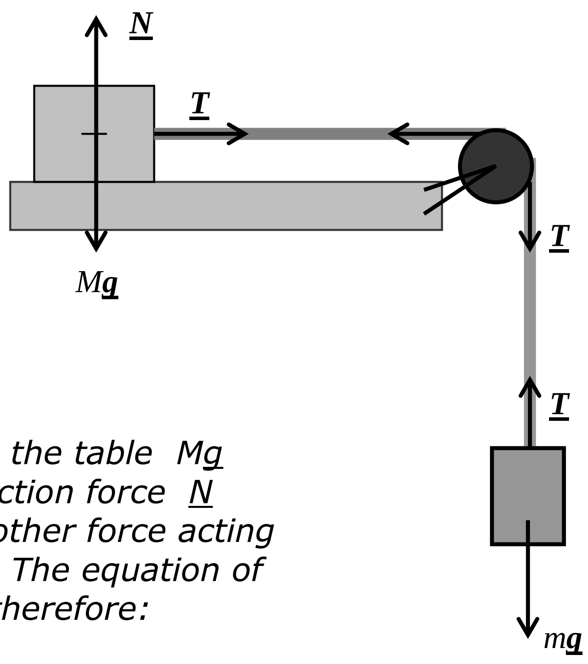
Since the acceleration is parallel to the slope we need the net force acting in this direction. This is the sum of the traction force F up the slope, the friction force f down the slope and the component of the weight, W.sin(θ), also down the slope. The equation of motion for the truck is therefore:

$$F - f - W\sin\theta = Ma \quad ,$$

so the acceleration is:

$$a = \frac{F - f - Mg\sin\theta}{M} = \frac{3000 - 200\times 1.6}{1.6\times 10^{3}} - 9.8\times\frac{1}{40} = 1.675 - 0.245 = 1.43\ m\,s^{-2}$$

Question 4p



A mass of 15 kg rests on a smooth table. This mass is connected by a light string, which passes over a light frictionless pulley, to another mass of 10 kg that hangs freely. Calculate the tension in the string and the acceleration of the system.

Solution: the weight of the mass on the table Mg is balanced by the perpendicular reaction force N as shown in the diagram. The only other force acting on M is the tension in the string, T. The equation of motion for the mass on the table is therefore:

$$T = Ma$$

(i)

where a is the acceleration. Obviously the acceleration of both masses must be the same, since they are connected by an inextensible string.

The net force on the other mass is its weight $m\underline{g}$ minus the tension in the string $\underline{T}$, so the equation of motion of this mass is

$$mg - T = ma$$

(ii)

Substituting for T from (i) gives:

$$mg - Ma = ma$$

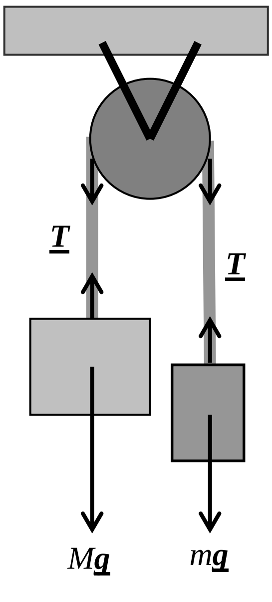
Rearranging we find the acceleration of both masses is

$$a = \frac{m}{m+M}g = \frac{10 \times 9.8}{10+15} = 3.92\ m\,s^{-2}.$$

The tension in the string, from (i), is: $T = Ma = 15 \times 3.92 = 58.8N$.

Question 4q

Two blocks, of masses 15 kg and 10 kg are connected together by a light string which passes over a light frictionless pulley as illustrated. Calculate the acceleration of the system and the tension in the string.



Solution: the tension in the string T is the same on both sides since there is no friction in the pulley. Assume that the mass M accelerates downward with acceleration a. The equation of motion for the mass M is then

$$Mg - T = Ma$$

(i)

The other mass will accelerate upwards at the same rate, so its equation of motion is:

$$T - mg = ma$$

(ii)

Adding equations (i) and (ii) together, we find

$$(M-m)g=(M+m)a$$

so the acceleration of the system is:

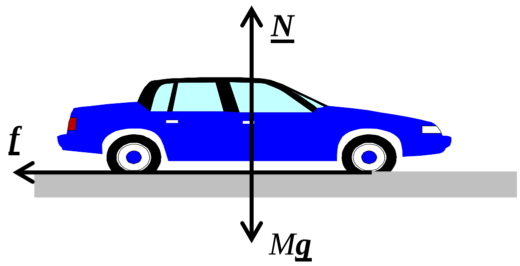
$$a=\frac{(M-m)}{(M+m)}g=\frac{15-10}{15+10}\times 9.8=1.96\ m\,s^{-2}.$$

The tension in the string is given by equation (i):

$$T=M(g-a)=15\times(9.8-1.96)=117.6\ N$$

Question 4r

The record for the longest skid mark on a public road was set by a Jaguar on the M1 in 1960. It was measured at 290 metres. Assuming the coefficient of kinetic friction between the wheels and the road $\mu_k = 0.6$ and that the deceleration was constant, how fast was the car travelling when its wheels became locked?



Solution: assuming that the clutch is depressed so the engine is disengaged, the only force affecting the horizontal motion of the car is the friction force f. The other forces are the weight of the car Mg and the normal reaction force N, which balance each other:

$$N=Mg \qquad (i)$$

The friction force is related to the normal reaction force via:

$$f=\mu_k N=\mu_k Mg \qquad (ii)$$

so the equation of motion of the car is:

$$-f=Ma \qquad (iii)$$

The minus sign here arises since the direction of the force is opposite to the direction of motion of the car. From (ii) the equation of motion becomes

$$-f = -\mu_k Mg = Ma.$$

The acceleration of the car is then $a = -\mu_k g = -0.6 \times 9.8 = -5.88\ m\,s^{-2}$.

To find the initial velocity we use the formula $v^2 = u^2 + 2as$. *Since the final velocity v = 0, we have*

$$u^2 = -2as = -2 \times (-5.88) \times 290 = 3410.4 (m\,s^{-1})^2$$

The initial velocity was therefore

$$u = \sqrt{3410.4} = 58.4\ m\,s^{-1} \quad \text{or} \quad 210\ km/hour.$$

Note that the length of the skid is proportional to the square of the velocity – clearly something to remember when estimating stopping distances.

Impulse

In all the preceding examples we have used the $F = ma$ form of Newton's second law. Let us consider the other form, where the force is equated to the rate of change of momentum. In the notation of differential calculus we may write:

$$\underline{F} = \frac{d\underline{p}}{dt} = \frac{d(m\underline{v})}{dt}$$

It follows that the integral of the force (with respect to time) is the change in the momentum:

$$\int_{t_1}^{t_2} \underline{F}\,dt = \underline{p}(t_2) - \underline{p}(t_1) = m\underline{v}_2 - m\underline{v}_1$$

If a constant force F acts for a time Δt, the integral is just $F\Delta t$. This is called the *impulse*. We can determine an impulse by measuring the change in the momentum of a body:

$$\boxed{\textit{Impulse = force} \times \textit{time = change in momentum}}$$

Question 4s

The force on a 10 kg object increases linearly from zero to 50 N in 4 seconds. What is the object's final speed if it starts from rest?

Solution: if the mass m starts from rest and reaches a speed v_f the change in its momentum is mv_f. The force increases linearly with time: so put

$$F(t) = bt$$

Then, since F = 50 N after 4 seconds we have $b = (50/4)\ N\,s^{-1}$. *From the definition of impulse:*

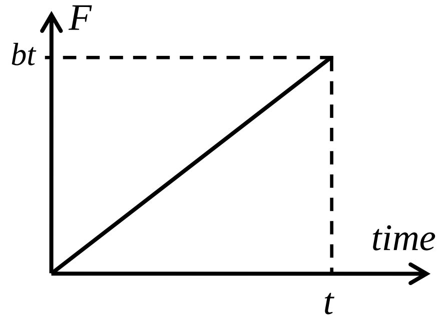
$$mv_f = \int_0^4 F(t)dt = \int_0^4 btdt = ½\,[bt^2]_0^4 = ½\left[\frac{50}{4}t^2\right]_0^4 = \frac{50\ \times 16}{2\ \times\ 4} = 100\ kg\ m\ s^{-1}$$

The final speed is then:

$$v_f = \frac{100\ kg\,m\,s^{-1}}{10\ kg} = 10\ m\,s^{-1}.$$

The impulse may be regarded as the area under the graph of F against t : in this example, where F = bt, the area is the area of the lower triangle in the diagram opposite. So,

$$\text{Impulse} = \tfrac{1}{2}bt \times t = \tfrac{1}{2}bt^2$$



Question 4t

A baseball of mass 0.15 kg, travelling at 40 m s^{-1}, is struck squarely by a bat, causing the baseball to rebound in the opposite direction at a speed of 60 m s^{-1}. The time of contact between the ball and bat is 1.3×10^{-3} seconds. Calculate the average force exerted by the ball on the bat.

First calculate Δp, the change in momentum of the ball:

$$\Delta \underline{p} = \underline{p}_f - \underline{p}_i = m(\underline{v}_f - \underline{v}_i)$$

It follows that

$$\Delta p = 0.15 \times (-60 - 40) = 15\ kg\,m\,s^{-1}$$

Since

$$F\,\Delta t = \Delta p$$

the average force is

$$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{15(kgm\,s^{-1})}{1.3\times10^{-3}(s)} = 1.15\times10^{4}\,N$$

Newton's Third Law and the Conservation of Momentum

Suppose two objects interact or collide with each other but are otherwise isolated. The only force on each body is due to its interaction with the other. Newton's third law says that the force on object 1 due to object 2 is equal and opposite to the force on object 2 due to object 1, i.e.

$$\underline{F}_{12} \;=\; -\,\underline{F}_{21}$$

Now according to the second law

$$F_{12} \;=\; \frac{d\left(m_1 \underline{v}_1\right)}{dt}$$

and

$$F_{21} \;=\; \frac{d\left(m_2 \underline{v}_2\right)}{dt}$$

It follows that

$$\frac{d\left(m_1 \underline{v}_1\right)}{dt} \;=\; -\;\frac{d\left(m_2 \underline{v}_2\right)}{dt}$$

or, equivalently,

$$\frac{d}{dt}\left(m_1 \underline{v}_1 \;+\; m_2 \underline{v}_2\right) \;=\; 0$$

This says that the rate of change of the total momentum is zero, i.e. the total momentum is constant:

$$m_1 \underline{v}_1 \;+\; m_2 \underline{v}_2 \;=\; constant$$

This result can be generalised to a system of many interacting objects. The principle oif the conservation of momentum applies to all types of interactions and it has the status of a fundamental principle of physics:

The total linear momentum of an isolated system is conserved