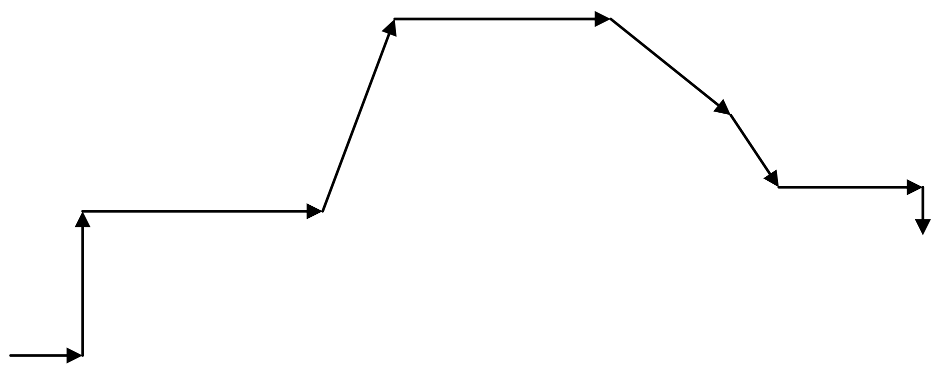
Section 3: Vectors, Forces and Static Equilibrium

In this section we will study how a body may remain in static equilibrium under the action of several forces. Forces are *vector* quantities, meaning that they have direction as well as a magnitude. To understand static equilibrium, therefore, we will need to know how to add forces together and resolve the action of a force onto a particular direction. To this end we start with a short tutorial on vectors.

3.1 Vectors

A scalar is a quantity that has a magnitude only: examples include mass, temperature, energy and so on. A vector is a mathematical representation of a quantity that possesses both a magnitude and a direction. There are many examples from everyday life where we could invoke a vector representation, for example a journey across the campus from the School of Physics and Astronomy to the Student's Union could be represented by a sequence of arrows:



Here the direction of the arrow gives the direction travelled in each section and the length of each arrow represents the distance travelled before a change of direction. Other examples of vector quantities include:

- Velocity – for example in weather forecasts the wind velocities across the UK are represented as a map covered with arrows, representing the strength and direction of wind.
- Momentum: this follows since momentum is the product of mass and velocity. Mass is a scalar, that is, it has only a magnitude. A vector multiplied by a scalar gives a vector.
- Acceleration: this is the rate of change of a velocity. Since velocity is a vector, then so is acceleration.

- Force: from Newton's law of motion, the action of a force induces an acceleration in a body:

  force = mass x acceleration

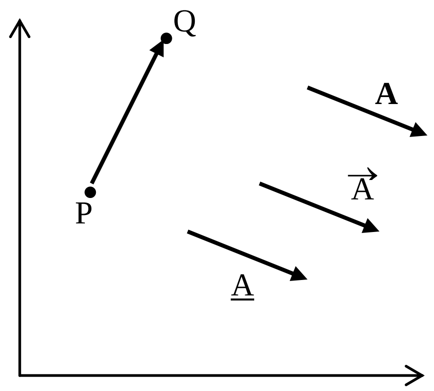
  Since acceleration is a vector, so is force.

How vectors are represented

We need two pieces of information to represent a vector, namely its direction and its magnitude. In the example we have already used, this information is conveyed by an arrow. The length of the arrow represents the magnitude of whatever physical quantity is represented by the vector. For example, in the journey across the campus the lengths would represent metres travelled, but on the weather map the lengths of the arrows would represent the wind speeds in metres per second. The length of a vector representing a force would be measured in newtons.

There are several different notation conventions for vectors. The most common are:

- $\overrightarrow{PQ}$ two capital letters with an arrow on top to indicate an arrow starting at point *P* and ending at point *Q*
- a bold faced letter, such as ***A***
- an underlined letter, such as $\underline{A}$ or $\underline{\boldsymbol{A}}$
- a letters with an arrow above: $\vec{A}$

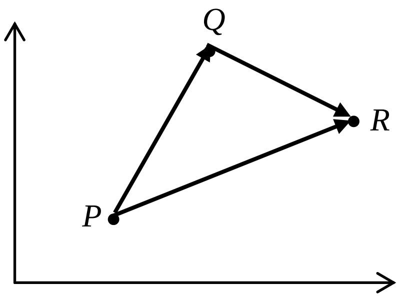


Note that a vectors may be drawn in any position (like the three versions of $\underline{A}$ in the diagram above) as long as its magnitude and direction is preserved.

The magnitude of a vector $\underline{A}$ is written as $|\underline{A}|$. The quantity $|\underline{A}|$ is therefore a scalar. Very often the magnitude is written simply as A to avoid complicated typography.

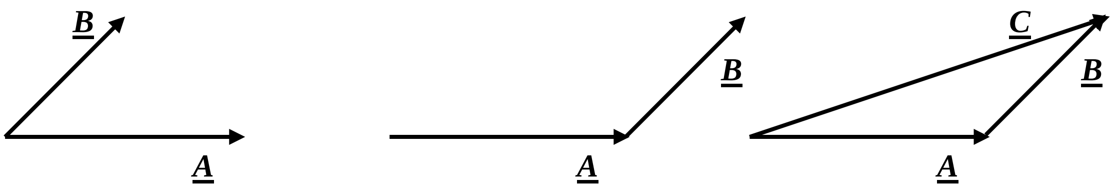
Adding vectors

The point *P* in the figure is first shifted to *Q* and then shifted to *R*. Clearly we could also start at the point *P* and shift it directly to *R*. The net effect is the same. We can interpret this equivalence as a vector addition:



$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

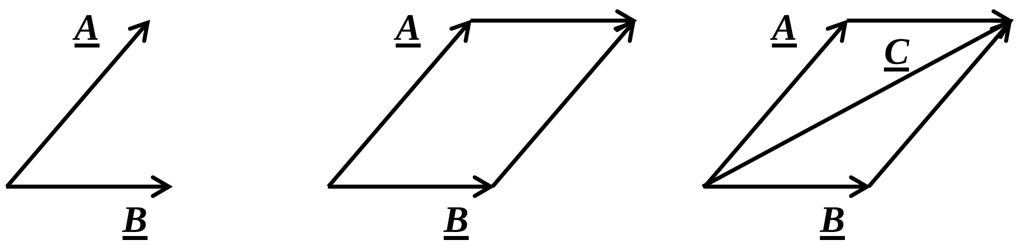
Consider two vectors $\underline{A}$ and $\underline{B}$ with a common origin as in the sketch (a) below. To add these together we can shift the vector $\underline{B}$ (without changing its magnitude or direction) until its starting point coincides with the end point of vector $\underline{A}$, as in (b). The vector sum of $\underline{A}$ and $\underline{B}$ is then the vector $\underline{C}$, which starts at the origin and ends on the end of vector $\underline{B}$, as in (c):



(a) (b) (c)

$$\underline{C} = \underline{A} + \underline{B}$$

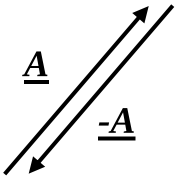
Notice that the sum of the two vectors can be represented as the diagonal of the parallelogram formed from $\underline{A}$ and $\underline{B}$ like this:



Subtracting vectors

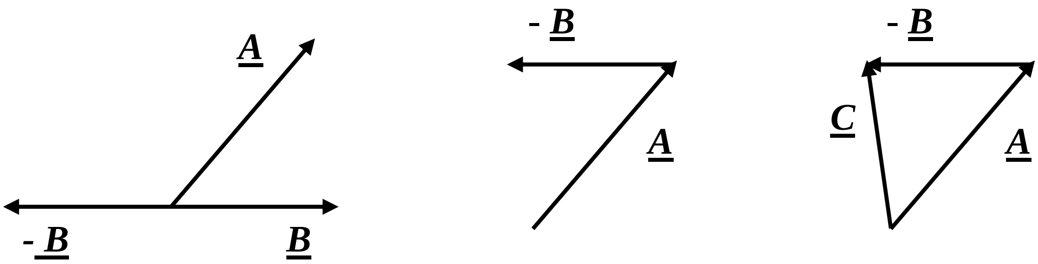
The negative of a vector $\underline{A}$ is a vector having the same magnitude but the opposite direction. We write it as $-\underline{A}$. Obviously the sum of a vector and its negative is zero:

$$\underline{A} + (-\underline{A}) = 0$$

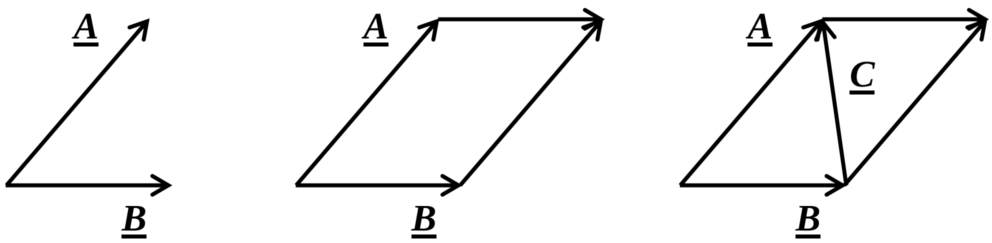


Subtracting $\underline{B}$ is equivalent to adding its negative, $-\underline{B}$:

$$\underline{A} - \underline{B} \;=\; \underline{A} + (-\underline{B}) \;=\; \underline{C}$$

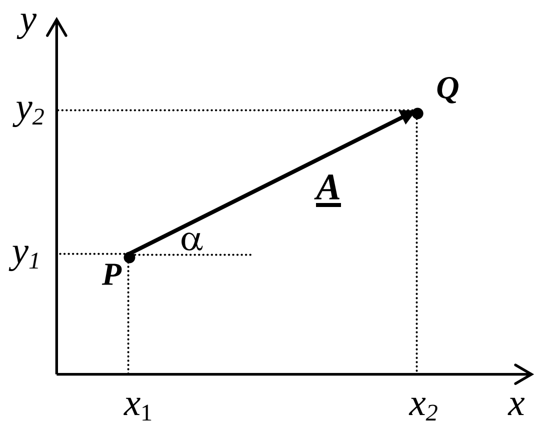


The difference between two vectors may also be constructed by again using the parallelogram constructed from $\underline{A}$ and $\underline{B}$, but now the difference vector $\underline{C}$ is the diagonal $\overrightarrow{BA}$:



Components and Projections of a Vector

Consider the vector $\underline{A}$ that shifts the point P (x_1, y_1) to the point Q (x_2, y_2) in the figure below. The displacement from P to Q corresponds to a shift of ($x_2 - x_1$) along the x axis and a shift of ($y_2 - y_1$) along the y axis. The shift along the x axis is called the *projection* of the vector on to the x axis, or the *x-component* of the vector. Similarly the shift along the y axis is called the *projection* of the vector on to the y axis, or the *y-component* of the vector.



The components are usually written as:

$$A_x = x_2 - x_1 \qquad\qquad A_y = y_2 - y_1$$

Suppose the vector $\underline{A}$ makes an angle α with the x axis, as in the figure above. Then simple trigonometry shows that

$$\cos(\alpha) = \frac{x_2 - x_1}{A} \qquad \text{and} \qquad \sin(\alpha) = \frac{y_2 - y_1}{A},$$

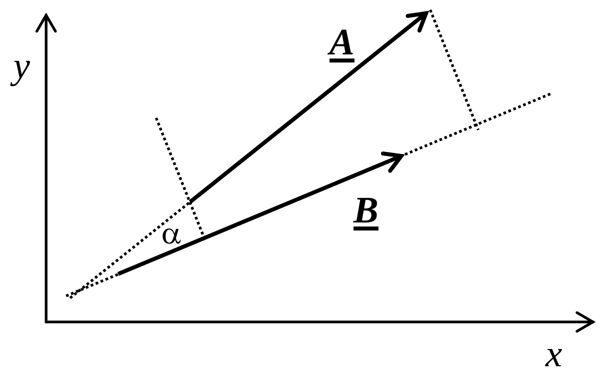
where, from the Theorem of Pythagoras, the modulus A of the vector $\underline{A}$ is given by:

$$A = \sqrt{A_x{}^2 + A_y{}^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can therefore write the components of the vector as:

$$A_x = A\cos(\alpha) \qquad\qquad A_y = A\sin(\alpha)$$

We can generalise the idea of projection by considering the projection of a vector onto an arbitrary direction, for example the projection of a vector $\underline{A}$ onto the direction of a vector $\underline{B}$:



All we need is the angle α between the two the two vectors, then the projection of $\underline{A}$ onto the direction of $\underline{B}$ is

$$A_B \quad = \quad A \cos(\alpha).$$

Similarly we can project the vector $\underline{B}$ onto the direction of the vector $\underline{A}$:

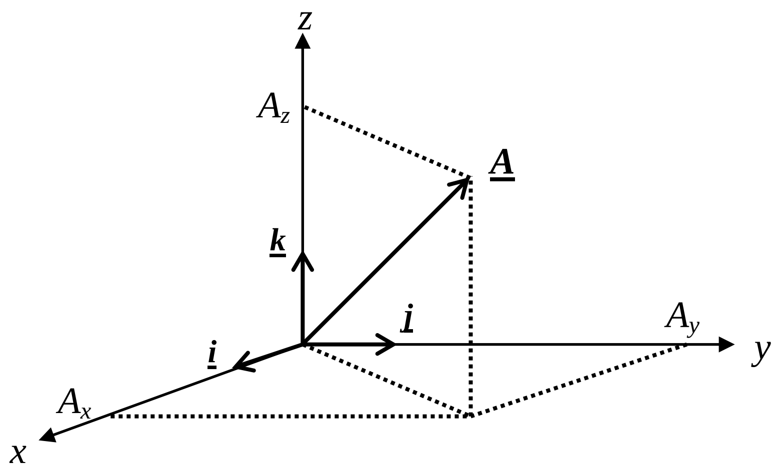
$$B_A = B \cos(\alpha).$$

Vectors in terms of Unit Vectors

A "unit vector" is defined as having a magnitude of one unit, so it just represents a direction. Unit vectors allow for a simple representation of a vector in terms of its components. In a Cartesian coordinate system (x, y, z), consider unit vectors in the directions of the three coordinate axes x, y and z. There are different notation conventions: these unit vectors could, for example, be variously denoted as

$$\underline{i},\ \underline{j},\ \underline{k} \qquad \text{or} \qquad \underline{\hat{x}},\ \underline{\hat{y}},\ \underline{\hat{z}} \qquad \text{or} \qquad \underline{e}_x,\ \underline{e}_y,\ \underline{e}_z$$

We shall use $\underline{i}$, $\underline{j}$, $\underline{k}$ below.



From the figure above we can see that the vector $\underline{A}$ may be written in terms of its *x, y,* and *z* components and the unit vectors as:

$$\underline{A} = A_x\, \underline{i} + A_y\, \underline{j} + A_z\, \underline{k}$$

Very often, a vector is written in terms of its components using brackets like this:

$$\underline{A} = (A_x, A_y, A_z)$$

Adding and subtracting vectors in terms of their components

To add or subtract two vectors we can simply add or subtract the individual components. For example, if

$$\underline{C} = \underline{A} + \underline{B} ,$$

then

$$\underline{C} = (C_x , C_y , C_z)$$

where

$$C_x = A_x + B_x , \qquad C_y = A_y + B_y , \qquad C_z = A_z + B_z .$$

Similarly, if $\underline{C} = \underline{A} - \underline{B}$,

Then $\underline{C} = (C_x , C_y , C_z)$

where $C_x = A_x - B_x$, $C_y = A_y - B_y$, $C_z = A_z - B_z$.

A couple of examples will help to make this clear:

(i) Let $\underline{A}$ = (2, 5, 1) and $\underline{B}$ = (3, -7, 4). Then the sum of these vectors is the vector

$$\underline{C} = \underline{A} + \underline{B} = (5, -2, 5).$$

(ii) The difference between the same two vectors is the vector

$$\underline{D} = \underline{A} - \underline{B} = (-1, 12, -3).$$

3.2 Forces as Vectors

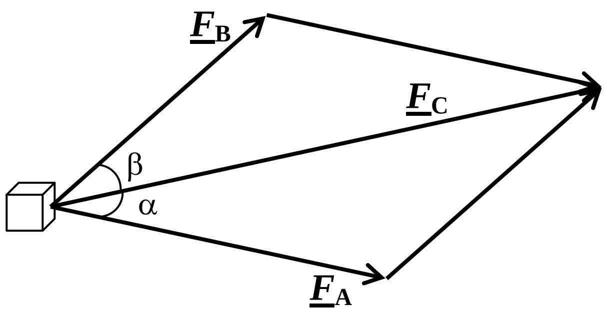
The resultant of two forces

Forces are vectors. If a body is subjected to two forces acting in different directions, then the net effect is equivalent to a single force equal to the vector sum of the two forces. This equivalent force is often called the *resultant* of the two forces. The resultant may be found in any of the following ways:

(i) The parallelogram of forces

Consider a body acted on by the two forces $\underline{F}_A$ and $\underline{F}_B$ as in the diagram below. We may find the resultant force $\underline{F}_C$ by constructing the parallelogram as shown:

$\underline{F}_C = \underline{F}_A + \underline{F}_B$



This is a graphical method requiring a ruler and a protractor: first the known vectors $\underline{F}_A$ and $\underline{F}_B$ are drawn, then $\underline{F}_C$ is constructed geometrically, then the magnitude and direction of $\underline{F}_C$ can be got by measurement.

(ii) Using components to find the resultant

Referring to the figure above, suppose we know that that the force $\underline{F}_A$ has components $(F_{A\,x}\,,\,F_{A\,y})$ and that the force $\underline{F}_B$ has components $(F_{B\,x}\,,\,F_{B\,y})$. It follows that the resultant $\underline{F}_C$ will have the components $(F_{C\,x}\,,\,F_{C\,y})$, where

$$F_{Cx} = F_{Ax} + F_{Bx} \qquad \text{and} \qquad F_{Cy} = F_{Ay} + F_{By}$$

This is true regardless of what the directions opf the coordinate axes (x and y) might be.

(iii) A way of finding the magnitude of the resultant

Suppose that in the diagram above the angle between the force $\underline{F}_A$ and the resultant $\underline{F}_{\mathbf{C}}$ is α, while the angle between the force $\underline{F}_B$ and the resultant $\underline{F}_C$ is β. We can find the magnitude of the resultant by resolving the forces $\underline{F}_A$ and $\underline{F}_B$ along the direction of $\underline{F}_C$:

$$F_C \;=\; F_A\cos(\alpha) \;+\; F_B\cos(\beta)$$

squaring this ,

$$F_C^2 \;=\; F_A^2\cos^2(\alpha) \;+\; 2F_AF_B\cos(\alpha)\cos(\beta) \;+\; F_B^2\cos^2(\beta)$$

We also know that the resultant has a zero component perpendicular to $\underline{F}_C$ (because the resultant **is** $\underline{F}_C$!). Algebraically, this means that

$$F_A\sin(\alpha) \;=\; F_B\sin(\beta)$$

or

$$F_A\sin(\alpha) \;-\; F_B\sin(\beta) \;=\; 0$$

and squaring this,

$$F_A^2\sin^2(\alpha) \;-\; 2F_AF_B\sin(\alpha)\sin(\beta) \;+\; F_B^2\sin^2(\beta) \;=\; 0$$

Adding this expression to the formula above for F_C^2, we get

$$F_C^2 \;=\; F_A^2 \;+\; F_B^2 \;+\; 2F_AF_B\,.\,(\,\cos(\alpha)\cos(\beta) \;-\; \sin(\alpha)\sin(\beta)\,)$$

using the trigonometrical identity $\sin^2(\theta) + \cos^2(\theta) = 0$. Now, the cosine of the sum of two angles is given by the standard trigonometrical formula

$$\cos(\alpha+\beta) \;=\; \cos(\alpha)\cos(\beta) \;-\; \sin(\alpha)\sin(\beta)$$

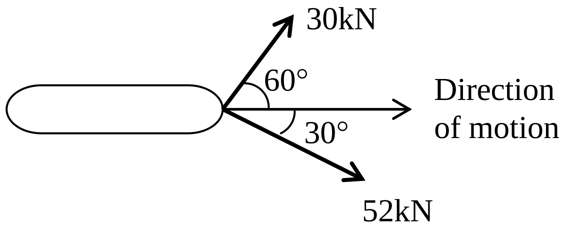
so we find

$$F_C^2 \;=\; F_A^2 \;+\; F_B^2 \;+\; 2F_A\,F_B\cos(\alpha+\beta)$$

This just amounts to an application of the "cosine rule" familiar in elementary trigonometry) to a triangle formed by the three forces F_A , F_B and F_C .

Question 3a

A ship is being towed by two tugs. The angles made by the two tow-ropes to the direction of motion of the ship and the magnitude of the tensions in the ropes are as shown in the diagram below. Calculate the resultant force acting on the ship.



Solution: taking components of the forces in the direction of motion of the ship, the magnitude of the resultant force is found to be

$F = 52\cos(30°) + 30\cos(60°)$ *(in kilonewtons)*

$= 52 \times 0.866 + 30 \times 0.5 = 45 + 15 = \underline{60\ \text{kN}}.$

Notice that we were actually supplied with more information than is necessary to answer this question. We have not used the fact that the two forces' components ***perpendicular*** *to the direction of motion must cancel out. Can you see that the question could still have been answered even if we had been told the magnitude of (say) only one of the forces?*

3.3 Static Equilibrium

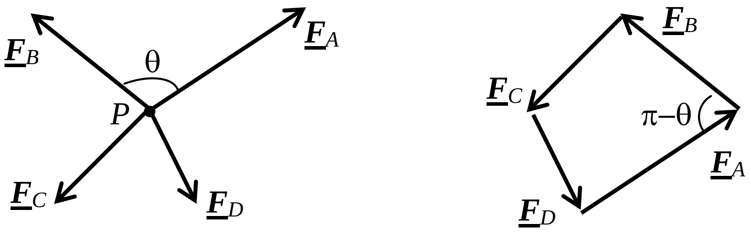
Newton's second law of motion tells us that a force F acting on a body of mass m will bring about an acceleration a of magnitude $a=F/m$. If a body is at rest, or moving with a constant velocity, its acceleration is zero and therefore it must have no net force acting on it. This could be because there is no force at all acting on the body, or it may be because there are two or more forces acting whose resultant vanishes. The body is then said to be in static equilibrium.

Expressing this algebraically, suppose that there are N separate forces act on a body at the same point. If the forces are represented by $\underline{\boldsymbol{F}}_1$, $\underline{\boldsymbol{F}}_2$, $\underline{\boldsymbol{F}}_3$, etc. then the body will be in static equilibrium if their vector sum vanishes:

$$\underline{\boldsymbol{F}}_1 + \underline{\boldsymbol{F}}_2 + \underline{\boldsymbol{F}}_3 + \underline{\boldsymbol{F}}_4 + \ldots + \underline{\boldsymbol{F}}_N = 0\ , \qquad \text{or} \quad \sum_{i=1}^{N} \underline{\boldsymbol{F}}_i = 0\ .$$

A polygon of forces

In a situation where a body is in static equilibrium under the influence of *N* coplanar forces acting at a point, the corresponding vector diagram must form a closed *N*-sided polygon: this condition satisfies the requirement that the resultant vanishes. In the example shown below, $N = 4$, so the four forces acting at a point on the left may be transformed into the closed quadrilateral on the right.

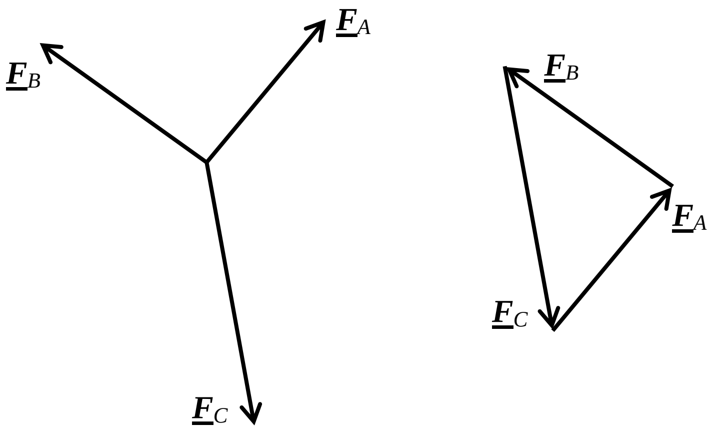


Note that the angle between the forces inside the polygon of forces is the *complement* of the angle between the vectors acting at the point. For example, in the diagram above the angle between the vectors $\underline{F}_A$ and $\underline{F}_B$ acting at *P* in the diagram above is the obtuse angle θ, while in the quadrilateral on the right the angle between the vectors $\underline{F}_A$ and $\underline{F}_B$ is the acute angle $(\pi - \theta)$.

A triangle of forces

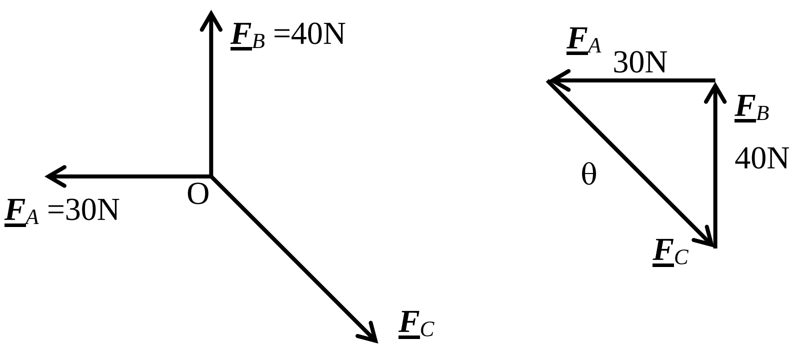
A number of practical problems involve static equilibrium under three coplanar forces acting at a point. The equilibrium condition is then met if the three forces form the sides of a triangle – this is just a special case of the polygon of forces discussed above.

An example of the triangle of forces is shown below: The three forces on the left must be in equilibrium since their vector sum forms the closed triangle on the right. Again note that the angles within the triangle are the complements of the angles between the forces on the left.



Question 3b

Three forces A, B and C act in the same vertical plane from a point O. Force A is 30 N and acts horizontally to the left of O. Force B is 40 N and acts vertically upwards. Determine the value of force C and the direction in which it acts, if the forces are in equilibrium.



Solution: the triangle of forces on the right is a right-angled triangle, so from Pythagoras' Theorem we have

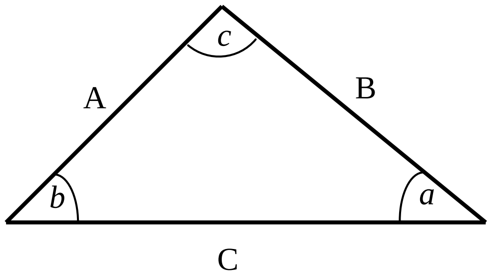
$$F_C = \sqrt{F_A{}^2 + F_B{}^2} = \sqrt{(30)^2 + (40)^2} = 50\ N$$

The angle θ *is given by* $\tan\theta = \frac{40}{30}$*, so that* $\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1°$*, so the force C is 50 N and acts to the right at an angle of 53.1° below the horizontal.*

A bit of trigonometry: the sine rule and the cosine rule

The "sine rule" and the "cosine rule" are two very useful equations from elementary trigonometry. They are stated here for reference.

For any triangle with sides A, B, C and corresponding angles *a,b,c* the sine rule states that



$$\frac{A}{\sin(a)} = \frac{B}{\sin(b)} = \frac{C}{\sin(c)}$$

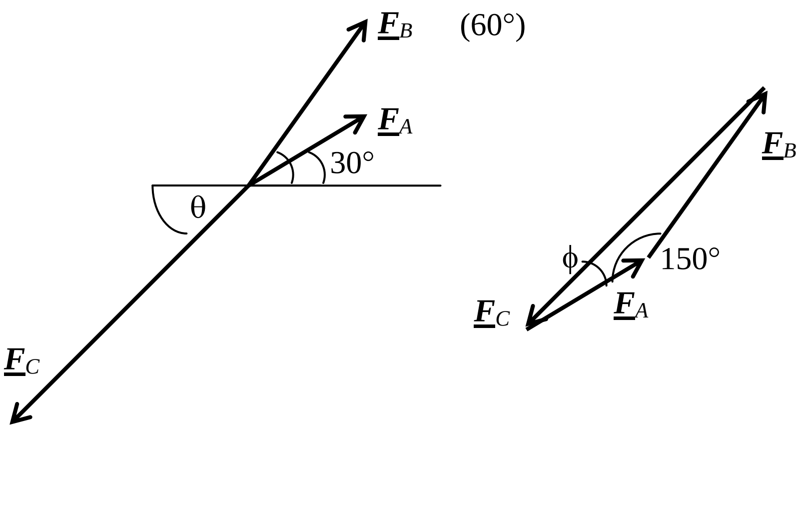
For the same general triangle the cosine rule is:

$$A^2 = B^2 + C^2 - 2BC\cos(a)$$

Question 3c

Calculate the force C needed to balance force A (which is 30 N at 30° to the x axis) and force B (which is 50 N at 60° to the x axis).

Solution: first draw a diagram:



The angle between A and B is 30°, so in the triangle of forces the obtuse angle between $\underline{F}_A$ and $\underline{F}_B$ is the complement of 30°, which is 150°. We can now use the cosine rule to find $\underline{F}_C$:

$$F_C^2 = F_A^2 + F_B^2 - 2F_A F_B \cos(c) = 30^2 + 50^2 - 2 \times 30 \times 50 \times \cos(150)$$

$$= 900 + 2500 - 3000 \times (-0.866) = 5998$$

so

$$F_C = 77.45 \text{ N}.$$

To find the direction of the force $\underline{F}_C$ we can use the sine rule: take the angle in the triangle of forces between $\underline{F}_A$ and $\underline{F}_C$ to be ϕ, then

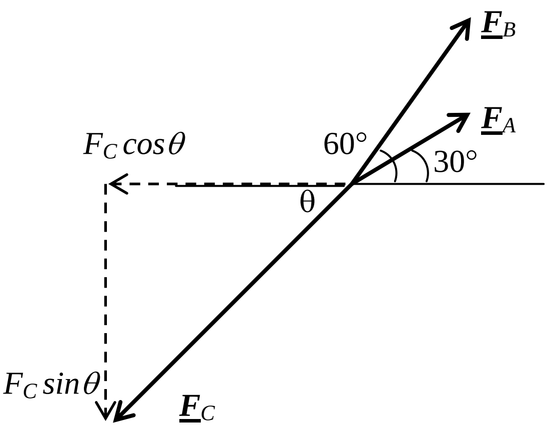
$$\frac{F_B}{\sin(\phi)} = \frac{F_C}{\sin(150)}$$

so that

$$\sin(\phi) = \frac{F_B \sin(150)}{F_C} = \frac{50 \times 0.5}{77.45} = 0.3228 .$$

It follows that ϕ = 18.83°, so that $\underline{F}_C$ is directed in the negative x direction at an angle θ = 18.83° + 30° = 48.83° below the axis.

Alternatively, we can solve the same problem by resolving the forces:



Resolving horizontally:

$$F_C \cos\theta = F_A \cos 30 + F_B \cos 60$$
$$= 30 \times 0.866 + 50 \times 0.5 = 50.98 \text{ N}$$

Resolving vertically:

$$F_C \sin\theta = F_A \sin 30 + F_B \sin 60$$
$$= 30 \times 0.5 + 50 \times 0.866 = 58.3 \text{ N}$$

By Pythagoras' Theorem we then have

$$F_C{}^2 \;=\; (F_c \sin\theta)^2 \;+\; (F_C \cos\theta)^2 \;=\; (58.30)^2 \;+\; (50.98)^2 \;=\; 5998\; N^2$$

so that F_C =77.45 N.

To find the direction of $\underline{F}_C$ we take the ratio of the two components:

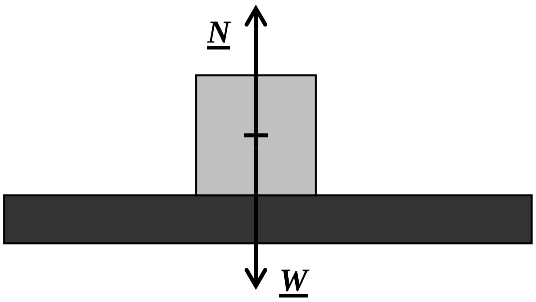
$$\frac{F_C \sin\theta}{F_C \cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{58.3}{50.98} = 1.1436$$

This gives

$$\theta \;=\; \tan^{-1}(1.1436) \;=\; 48.8°$$

Some particular examples of static equilibrium:

(i) Weight and Normal Reaction Force



Consider a block of mass M at rest on a surface. The block is in static equilibrium under the influence of two equal and opposite forces, namely, (a) the weight $\underline{W} = M\underline{g}$, due to the force of gravity acting on the mass M, and (b) the normal reaction force $\underline{N}$ due to the surface pushing back on the block.

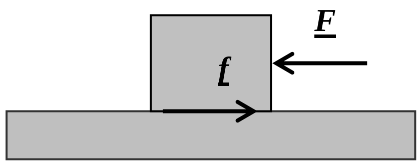
The reaction force arises from the slight deformation of the surface due to the weight of the block. The net force must be zero since the block is in equilibrium, so that

$$\underline{N} = -\underline{W},$$

so the magnitude of the normal reaction force is $N = Mg$.

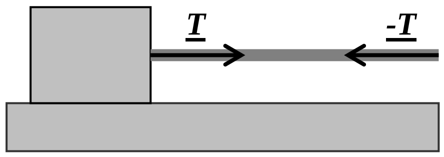
In general, when a body is pressed against a surface, the surface pushes back on the body with a reaction force that is perpendicular (normal) to the surface.

(ii) Frictional forces



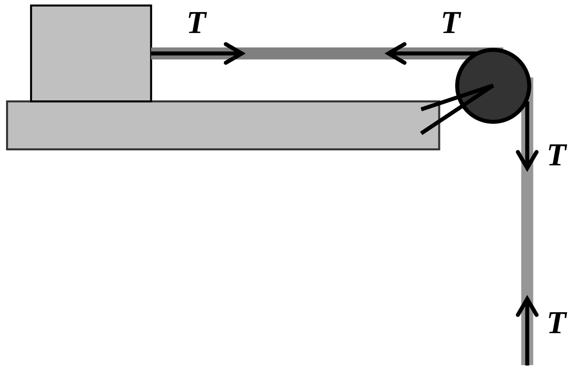
Suppose the block is subjected to a small sideways force $\underline{F}$. If the contact between the block and the surface is rough, there will be a frictional force $\underline{f}$ opposing the tendency of the block to slide across the surface. As long as the block is stationary, it must be in equilibrium, so the frictional force must exactly balance the applied force: $\underline{F} = -\underline{f}$. As the applied force is increased, eventually there will come a point where the block begins to move, since there is a limit to the magnitude of the frictional force. If the friction is negligible the contact between the block and the surface is said to be smooth or frictionless.

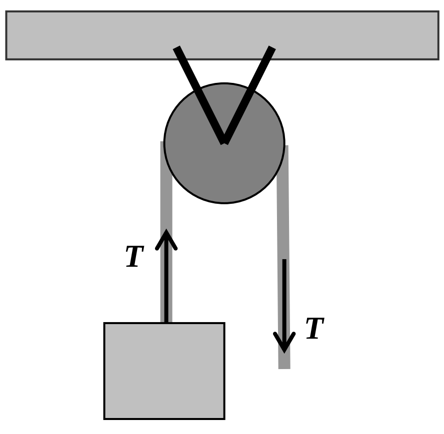
(iii) Tension in a rope, cord or cable



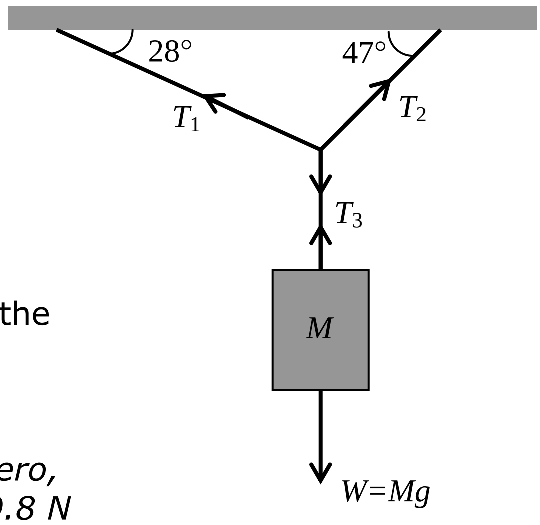
A force may be applied to a body by attaching a cord to it and pulling on the other end of the cord. The tension force T in the cord is transmittted along its length, so that while the body is pulled with a force $\underline{T}$ in the direction of the cord, the person pulling on the cord feels an equal and opposite force $-\underline{T}$ towards the body. In most situations we may ignore the mass of the cord and it may assumed to be unstretchable.

Some similar situations are illustrated below:





Question 3d



A block of mass M = 15 kg hangs from a cord attached by a knot to two other cords which are in turn fixed to the ceiling as shown in the sketch.

The cords have negligible mass. What are the tensions in the three cords?

Solution: firstly, since the block is in static equilibrium, the net force on the block is zero, so that $\underline{T_3} = \underline{W}$, i.e. $T_3 = Mg = 15 \times 9.8$ N = 147 N.

Next, the knot is in equilibrium, so that the three tensions must add vectorially to zero:

$$\underline{T_1} + \underline{T_2} + \underline{T_3} = 0$$

Firstly we resolve the forces horizontally: $T_{1x} + T_{2x} + T_{3x} = 0$. Using the data from above this yields

$$-T_1 \cos(28°) + T_2 \cos(47°) + 0 = 0 \qquad \text{(i)}$$

Now resolving vertically, $T_{1y} + T_{2y} + T_{3y} = 0$, whence

$$T_1 \sin(28°) + T_2 \sin(47°) - T_3 = 0 \qquad \text{(ii)}$$

We now have two simultaneous equations (i) and (ii) to determine the two unknowns T_1 and T_2. From (i) we can express T_1 in terms of T_2:

$$T_1 = T_2 \frac{\cos(47°)}{\sin(28°)} = 0.7724\, T_2$$

Substituting this value into (ii) along with T_3 = 147 N, we have:

$$0.7724\, T_2 \sin(28°) + T_2 \sin(47°) - 147 = 0$$

This becomes:

$$T_2\, (0.7724 \times 0.4695 + 0.7314) = 147$$

so that

$$T_2 = \frac{147}{1.094} = 134.4N$$

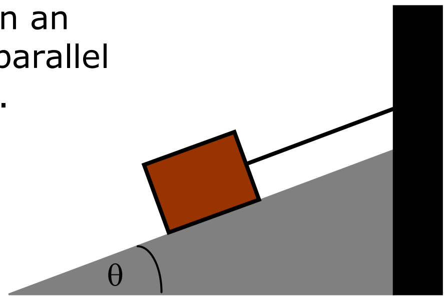
and

$$T_1 = 0.7724T_2 = 103.8N$$

The tensions in the cords are therefore $T_1 = 103.8$ *N,* $T_2 = 134.4$ *N and* $T_3 = 147$ *N.*

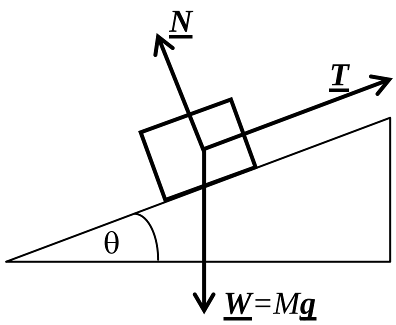
Question 3e

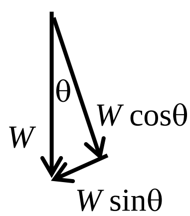
A crate of mass 15kg is held stationary on an smooth ramp, inclined at 27°, by a rope parallel to the plane of the ramp, as in the figure.



Calculate the values of the tension in the rope and the normal reaction force on the block.

Solution: we are told that the ramp is smooth, so we can ignore friction. The three forces under which the crate is in equilibrium are gravity acting on the mass, to give the weight $\underline{W}$*, the tension in the rope* $\underline{T}$ *and the normal reaction force* $\underline{N}$ *of the ramp on the crate, as shown in the sketch:*





Since T and N are respectively parallel and perpendicular to the slope of the ramp, it makes sense to resolve the forces along these directions. In the lower sketch we have resolved the weight into its two components, namely W.sin(θ) parallel the slope and W.cos(θ) perpendicular to the ramp.

It is clear, then, that for static equilibrium the normal reaction force is

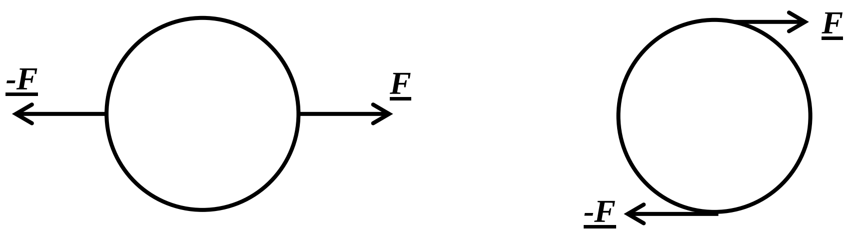
$$N = W\cos\theta = Mg\cos\theta = 15\times 9.8\times\cos(27) = 131N$$

and the tension in the rope is

$$T = W\sin\theta = Mg\sin\theta = 15\times 9.8\times\sin(27) = 66.7N$$

3.4 The moment of a force (torque)

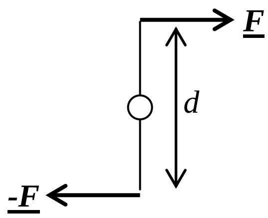
So far we have considered static equilibrium under a number of forces acting at the same point on a body. What happens if the forces do not act at a point? Consider, for example, the two discs below :



The disc on the left is in static equilibrium under two equal and opposite forces *F* and -*F* acting through a point (the centre of the disc). The disc on the right is *not* in static equilibrium, since in that case the two forces do not act through the same point and the disc will therefore rotate under the action of the forces. A pair of forces separated by a perpendicular distance *d* are sometimes called a *couple.* The magnitude of the torque, Γ, can be defined as the product of the force *F* and the perpendicular distance *d*:

$$\Gamma = Fd$$

Such a torque will lead to rotation about the midpoint O.



The torque due to any force around any point is the product of the force and the perpendicular distance from the point to the line of action of the force. Note that, instead of just multiplying the value of each force by the distance between them:

$$\Gamma = Fd$$

it is slightly more correct to regard each force, F and -F. as contributing half the torque, writing

$$moment\ of\ forces = F\frac{d}{2} + F\frac{d}{2} = Fd = \Gamma\ .$$

Here, *d/2* is obviously the perpendicular distance from the central axis to the line of action of either force.

Conditions for the static equilibrium of a body:

For static equilibrium we have to impose the condition that the sum of all the external torques acting on the body must be zero. So we have:

1. The vector sum of all the external forces acting on the body must be zero.
2. The sum of all the external torques acting on the body must be zero, measured about *any* point.

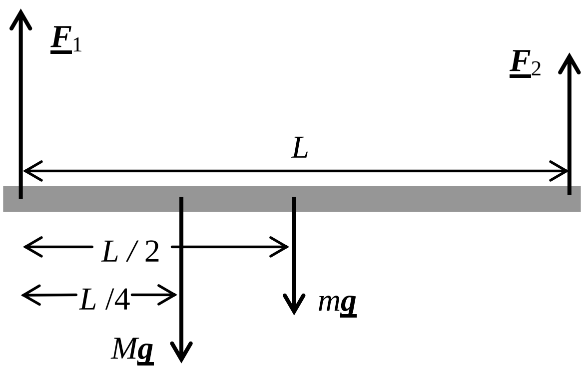
The first of these conditions prevents any linear motion of the body, and the second prevents any rotational motion of the body.

Question 3f

The ends of a uniform beam of mass *m* = 1.8 kg are supported on two scales. A block of mass *M* = 2.7 kg rests on the beam with its centre one-quarter of the way along the beam from the left-hand end, as shown in the diagram. What are the readings on the two scales?

Solution: the sketch below gives a summary of the forces acting on the beam. There are no horizontal components to worry about. The vertical components must balance if the beam is in static equilibrium:

$$F_1 + F_2 = Mg + mg \qquad (i)$$



There are two unknowns, F_1 and F_2, so we must use the second stability criterion to solve the problem: we take moments of the forces about the left hand end of the beam. Since F_1 passes through this point it will make no contribution to the moments. The clockwise moments will add up to:

$$(Mg)(L/4) + (mg)(L/2)$$

while just one force provides an anticlockwise moment which is

$$(F_2)(L) \ .$$

Equating the clockwise and anticlockwise moments we get

$$F_2 L = (Mg)(L/4) + (mg)(L/2)$$

(ii)

(Strictly speaking, the clockwise moment should be regarded as having a negative sign and what we are doing is setting the total moment equal to zero.)

It follows that

$$F_2 = (Mg/4) + (mg/2) = (2.7/4 + 1.8/2) \times 9.8 = 15.44N.$$

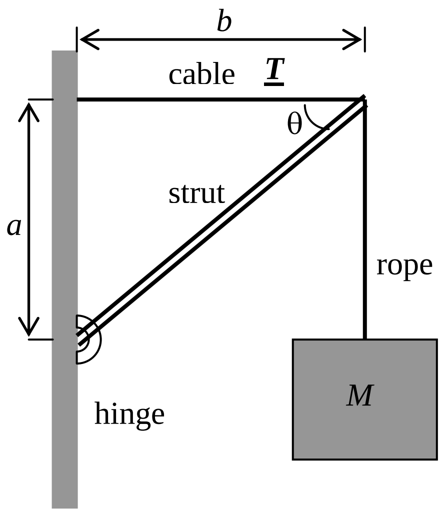
We now substitute this value into eq. (i) above:

$$F_1 = Mg + mg - F_2 = (2.7 + 1.8) \times 9.8 - 15.44 = 28.67N.$$

Actually we could have taken moments about any point along the beam. Whichever point was chosen, the same answers would have been arrived at. By selecting one end we have kept the algebra as simple as possible.

Question 3g

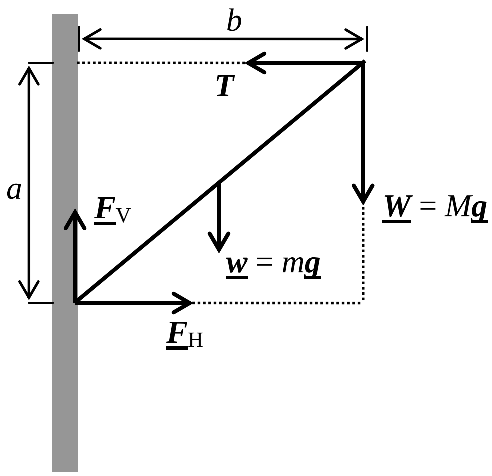
A safe of mass 430 kg hangs by a rope from a hinged strut. The strut is maintained at an angle θ to the horizontal by a cable attached to the wall, vertically above the hinge, as shown in the diagram.



The mass of the strut is 85 kg and the dimensions are a = 1.9 m and b = 2.5 m. The mass of the cable and rope are negligible.

Calculate (i) the tension in the cable and (ii) the reaction force at the hinge.

Solution: the forces acting are shown in the sketch below.



The reaction force at the hinge is taken to have a horizontal component F_H and a vertical component F_V. In order to calculate the tension in the cable we can take moments of the forces about the hinge. In this case we don't have to bother about the reaction force since it passes through the hinge and so makes zero contribution to the moments.

The clockwise moments due to the weight of the beam and the weight of the safe are:

$$w \times (b/2) + W \times (b) = (m/2 + M) \times (bg)$$

The anticlockwise moment due to the tension in the cable is $T \times a$.
Equating the clockwise and anticlockwise moments gives

$$T = (m/2 + M) \times (b/a) \times g = (85/2 + 470) \times (2.5/1.9) \times 9.8 = 6092.8N$$

Now, to find the reaction forces at the hinge we resolve the forces horizontally and vertically. Horizontally:

$$F_H = T = 6092.8N$$

Vertically:

$$F_V = W + w = (M + m)g = (430 + 85) \times 9.8 = 5047N$$

Using Pythagoras' Theorem, the magnitude of the resultant reaction force is found to be

$$\sqrt{(6093)^2 \;+\; (5047)^2} \;=\; 7812\,N$$

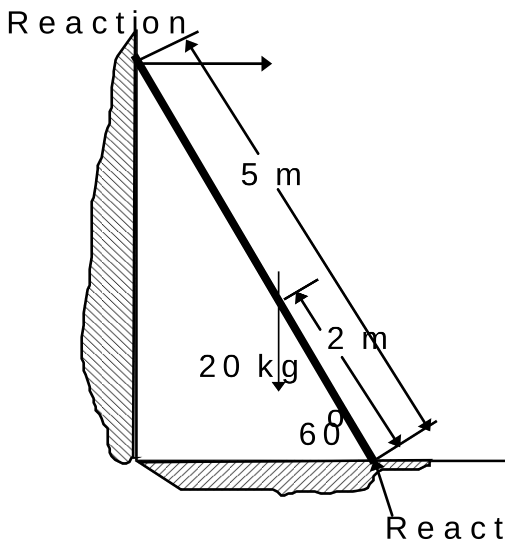
and the force is directed at an angle

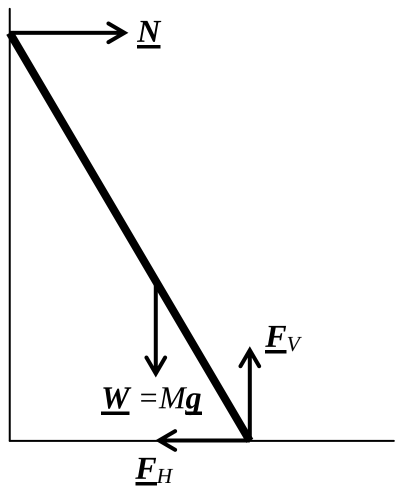
$$\tan^{-1}(F_V / F_H) = \tan^{-1}(5047/6092.8) = 39.6\,^{\circ}$$

to the horizontal.

Question 3h

A ladder is resting against a wall as shown with its upper end against a smooth wall and its lower end supported on rough ground. The ladder has a mass of 20 kg and its weight may assume to act through a point 2 m from the lower end. Determine the reaction forces of the wall and the ground on the ladder.





Solution: the forces are sketched in the diagram on the right, where we have resolved the reaction force at the ground into a horizontal component $\underline{F}_H$ *and a vertical component* $\underline{F}_V$*. We are told in the problem that the wall is smooth: this means there is no friction, so the reaction force at the wall* $\underline{N}$ *must be perpendicular to the wall.*

The first step is to find the value of N. To do this we can take moments of the forces about the point of contact of the ladder with the ground. This eliminates the reaction forces at the ground. Taking the ladder to be of length L, the clockwise moment is due to the reaction force $\underline{N}$ *: this has magnitude* $(N)\times(L\sin(60))$*. Here* $L\sin(60)$ *is the perpendicular distance between the point of contact with the ground and the line of action of the normal reaction force* $\underline{N}$*.*

The anticlockwise moment is due to the weight of the ladder: this has a magnitude $(Mg)\times(2L/5)\times\cos(60)$*. Again* $(2L/5)\times\cos(60)$ *is the perpendicular distance between the point of contact with the ground and the line of action of the weight* $\underline{W}$*.*

Equating the clockwise and anticlockwise moments of the forces gives:

$$(N)\times(L\sin(60)) = (Mg)\times(2L/5)\times\cos(60)$$

It follows that:

$$N = Mg\times\frac{2\cos(60)}{5\sin(60)} = 20\times 9.8\times\frac{2\times 0.5}{5\times 0.866} = 45.27N$$

Now, to work out the components of the reaction force at the ground, we resolve the forces horizontally and vertically. Resolving horizontally gives: $F_H = N = 45.27N$*, while resolving vertically gives*

$$F_V = Mg = 20\times 9.8 = 196N$$

The resultant reaction force is therefore

$$F = \sqrt{F_H{}^2 + F_V{}^2} = \sqrt{(45.27)^2 + (196)^2} = 201\,N$$

at an angle

$$\tan^{-1}(F_V/F_H) = \tan^{-1}(196/45.27) = 77^\circ$$

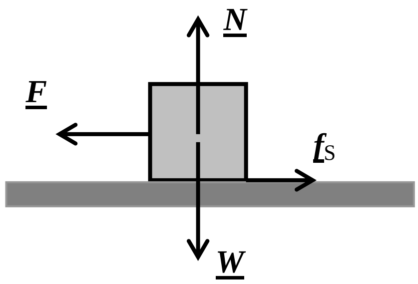
to the horizontal.

3.5 Friction

Static equilibrium and all motion are strongly affected by friction between surfaces. The concept of "smooth", i.e. frictionless, surfaces is an idealisation, useful as an approximation when solving problems, but not achievable in practice. Friction is, of course, essential for many aspects of everyday life, since it provides traction when we walk or drive a car or bicycle along the road. It would be impossible to erect a ladder against a wall without friction between the ladder and the ground.

Friction between relatively smooth surfaces arises from adhesion at a microscopic scale. The actual microscopic area of contact between two surfaces is limited to the "peaks" which might constitute perhaps only 10^{-4} of the apparent contact area. This means that the local pressures at these contact points are 10^4 times greater than the nominal load per unit area. Such pressures can cause "cold welding" at the contact points. To produce relative motion between the two surfaces this adhesion has to be overcome: the force required is what we experience as frictional force.

If two surfaces are pressed harder together, many more contact points cold-weld, so it requires a greater applied force to get the surfaces to slide relative to each other. This means that the static frictional forces increase as the load on the surfaces is increased.



Consider applying a sideways force $\underline{F}$ to a block lying on a surface. For a small force the block will remain in static equilibrium, since the static frictional force $\underline{f}_S$ exactly balances the applied force. As the applied force is increased the magnitude of the static frictional force also increases so that the block remains at rest. Eventually the applied forceovercomes the adhesion between the surfaces and the block breaks away and accelerates in the direction of the applied force. The frictional force that opposes the motion of the moving block is called the kinetic frictional force. The magnitude of the kinetic friction is lower than the maximum value of the static friction, but still increases with the load. Kinetic friction leads to "jerky" motion, since it results from the momentary forming and breaking of adhesive cold-welds between the surfaces. We are often made aware of the jerkiness through the sound produced, for example the squeak of a finger-nail scratching a blackboard, or the sound of a violin, produced by drawing the bow across the strings.

Properties of friction

There are a number of empirical rules relating to friction:

(i) If a body is in static equilibrium under the application of a force, then the static friction f_S exactly balances the component of the applied force that is parallel to the surface. The limiting value of the static frictional force $f_{S,\max}$ is proportional to the magnitude of the normal reaction force on the body from the surface, i.e.

$$f_{S,\max} = \mu_S N$$

where μ_S is called the *coefficient of static friction*.

(ii) If the body begins to slide along the surface, the magnitude of the frictional force drops below $f_{S,\max}$ to a lower value (the "kinetic" frictional force) f_K which is also proportional to the magnitude of the normal reaction force:

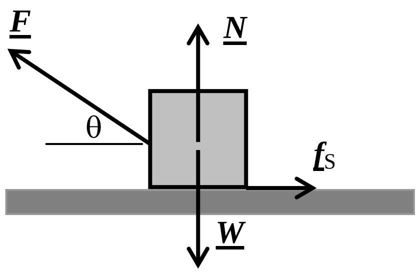
$$f_K = \mu_K N \, ,$$

where μ_K is called the *coefficient of kinetic friction*.

(iii) The frictional force is independent of the area of contact.

Limiting friction on a horizontal plane

Consider a heavy block of mass M at rest on a horizontal surface. Suppose a force F is applied at an angle θ to the horizontal. The force increases in magnitude until the block is just on the point of moving. At this point we know that the static frictional force has a magnitude $f_{S,\max} = \mu_S N$.



First we resolve the forces in the problem horizontally:

$$F\cos(\theta) = f_{S,\max} = \mu_S N \, . \qquad \text{(i)}$$

Then vertically:

$$F\sin(\theta) + N = W = Mg \qquad \text{(ii)}$$

From (ii) we have

$$N = Mg - F\sin(\theta)\,.$$

Substituting for *N* in (i) we find#

$$F\cos\theta \;=\; \mu_S\,(Mg \;-\; F\sin\theta)$$

Rearranging, we find that the minimum force required to move the block is

$$F \;=\; \frac{\mu_S Mg}{(\cos\theta \;+\; \mu_S\sin\theta)}$$

Question 3i

A metal block of mass 100 kg is to be dragged across a rough floor using a rope inclined at 30°. If the coefficient of static friction is 0.75, what is the smallest force required to move the block?

Solution: using the result above,

$$F \;=\; \frac{\mu_S Mg}{(\cos\theta \;+\; \mu_S\sin\theta)} \;=\; \frac{0.75\;\times 100\;\times 9.8}{(\cos 30° \;+\; 0.75\;\sin 30°)} \;=\; 593\,N$$

Optimal angle: the "angle of friction"

Clearly we have a choice of the angle at which to apply the external force. If we pull at an angle θ to the horizontal, the net normal force is reduced by the vertical component of the external force. This reduces the limiting static friction, and hence the magnitude of the smallest force required to move the block. However if the angle is too steep, the horizontal component of the external force will not be sufficient to overcome the friction. Given the weight of the block and the coefficient of friction, can we calculate the optimal angle at which to apply the force in order to move the block with least effort?

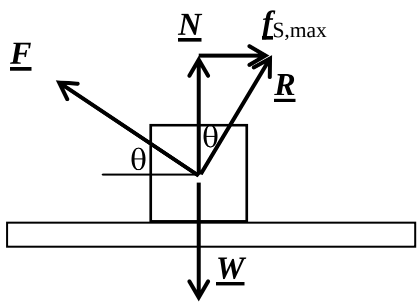
From the previous result the minimum value of the external force will occur when the quantity $(\cos\theta + \mu_S \sin\theta)$ is at a maximum. A little bit of mathematics shows that this condition is met when

$$\theta = \tan^{-1}(\mu_S)$$

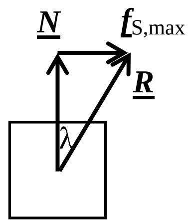
The construction below shows that the resultant $\underline{R}$ of the normal reaction force $\underline{N}$ and the frictional force $\underline{f_S}$ makes an angle of

$$\theta = \tan^{-1}(f_{S,\max} / N) = \tan^{-1}(\mu_S)$$

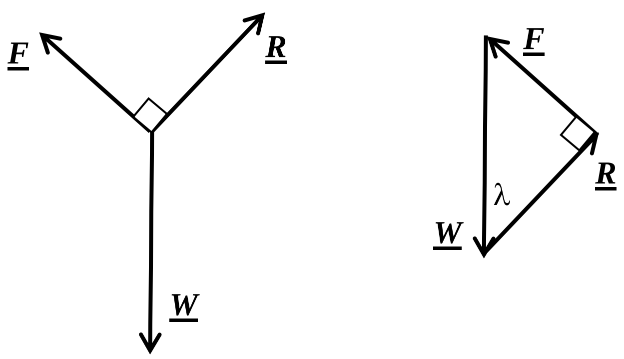
to the normal, so that the optimal angle for the external force is at right angles to $\underline{R}$.



This construction defines the *angle of friction,* $\lambda = \tan^{-1}(\mu_S)$.



Using the resultant $\underline{R}$ and the angle of friction we can reduce the problem to a triangle of forces, as shown in the following diagram.



Question 3j

A metal block of mass 100 kg is to be dragged across a rough floor using a rope, as in question 3g, but this time we can choose the inclination of the rope. If the coefficient of static friction is 0.75, what is the smallest force required to move the block?

Solution: from the construction, and remembering that $\underline{F}$ *and* $\underline{R}$ *are at right angles to each other we can use the triangle of forces to find F directly:* $F = W\sin\lambda$*. But the angle of friction*

$$\lambda \quad = \quad \tan^{-1}(\mu_S) \quad = \quad \tan^{-1}(0.75) \quad = \quad 37°$$

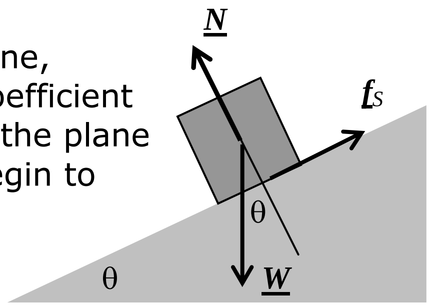
so

$$F_{\min} = Mg\sin\lambda = 100 \times 9.8 \times \sin(36.9^{\circ}) = 588N$$

This value is indeed smaller than the force calculated earlier.

Limiting friction on an inclined plane

Consider a block of mass *M* resting on a plane, inclined at angle θ to the horizontal. The coefficient of friction is μ_S. Suppose that the angle of the plane is increased: at what angle will the block begin to slide down the plane?



Initially the block is in equilibrium under the action of the weight, $\underline{\boldsymbol{W}}$, the normal reaction force, $\underline{\boldsymbol{N}}$, and the frictional force $\underline{\boldsymbol{f}}_S$. As the plane tilts further the component of the weight parallel to the plane will increase until the frictional force reaches its limiting value $\underline{\boldsymbol{f}}_{S,max}$. Let us resolve the forces parallel and perpendicular to the slope.

Resolving perpendicular to the plane:

$$W\cos\theta = N \qquad \text{(i)}$$

Resolving parallel to the plane:

$$W\sin\theta = f_{S,\max} = \mu_S N \qquad \text{(ii)}$$

Using (i) we can eliminate the normal reaction force *N*. Then (ii) becomes

$$W\sin\theta = f_{S,\max} = \mu_S W\cos\theta$$

so that the limiting angle is determined by

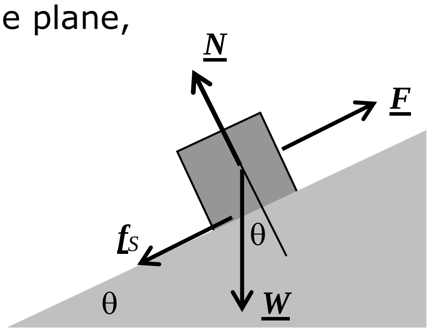
$$\frac{W\sin\theta}{W\cos\theta} = \tan\theta = \mu_S$$

i.e. $\theta = \tan^{-1}(\mu_S)$. This is identical to the angle of friction λ.

This result gives a very simple way of determining the coefficient of friction between a body and a surface: just tilt the surface until the body is on the verge of slipping, then the tangent of the slope gives a value for the coefficient of friction.

Question 3k

A metal block of mass 100kg rests on a plane surface inclined at 20° to the horizontal. The coefficient of friction is 0.75. A force is applied to the block with a rope parallel to the slope of the plane. What is the smallest force required to move the block (a) up the plane, and (b) down the plane?



Solution:

(a) We resolve the forces parallel and perpendicularly to the plane. Resolving perpendicularly,

$$N = W\cos\theta \quad \text{(i)}$$

Resolving parallel to the plane:

$$F = W\sin\theta + f_{S,\max} = Mg\sin\theta + \mu_S N \qquad \text{(ii)}$$

Substituting the value of N from (i) gives

$$F = W\sin\theta + \mu_S N = W\sin\theta + \mu_S W\cos\theta = Mg(\sin\theta + \mu_S\cos\theta)$$

$$= \quad 100 \times 9.8 \times (0.342 + 0.75 \times 0.9397) = 1026\,N$$

(b) In the second case the force F is directed down the slope and the frictional force acts up the slope, to oppose the motion. Resolving the forces perpendicular to the slope gives the same equation (i) as before. Resolving parallel to the slope gives:

$$F + W\sin\theta = \mu_S N = \mu_S W\cos\theta\,.$$

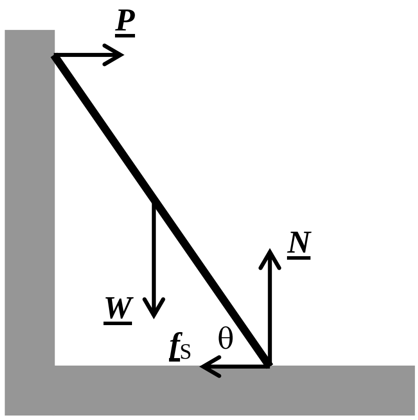
Rearranging gives

$$F = \mu_S W\cos\theta - W\sin\theta = Mg(\mu_S\cos\theta - \sin\theta)$$
$$= 100\times 9.8\times(0.75\times 0.9397 - 0.342) = 355.5N$$

Question 3l

A uniform ladder, of length L, rests against a smooth wall at an inclination θ to the floor. The coefficient of friction between the ladder and the floor is μ_S. Show that the shallowest angle at which the ladder is stable is given by the equation

$$\theta = \tan^{-1}\left(\frac{1}{2\mu_S}\right).$$



Solution: the forces in the problem are as shown in the sketch. We have labelled the reaction force at the wall P. Since the wall is smooth, i.e. frictionless, this reaction force is normal to the wall. The reaction force at the floor has been resolved into a normal component N and the frictional force f_S. The shallowest angle for stability of the ladder will be determined by the maximum value that the frictional force can take, namely $f_{S,max}$, which of course is determined by the coefficient of friction and the normal reaction force N.

First calculate P by taking moments of the forces about the point of contact with the floor: this takes both f_S and N out of the equation.

$$\textit{Clockwise moment } = (P)\times(L\sin\theta)$$

$$\textit{Anticlockwise moment } = (W)\times(L/2)\times\cos\theta$$

Equating these we find

$$P = W \times \frac{L\cos\theta}{2L\sin\theta} = \frac{mg}{2\tan\theta} \qquad \textit{(i)}$$

Now resolving the forces horizontally:

$$P = f_S, \qquad \textit{(ii)}$$

and resolving vertically:

$$N = W = mg. \qquad \textit{(iii)}$$

The limiting case is when $f_S = f_{S,\max} = \mu_S N$. *It follows from (iii) that:*

$$f_{S,\max} = \mu_S N = \mu_S mg,$$

and from (i) and (ii)

$$P = f_{S,\max} = \mu_S mg = \frac{mg}{2\tan\theta}.$$

This yields the required result, namely that the smallest angle for stability of the ladder is given by

$$\tan\theta = \frac{1}{2\mu_S}$$

so

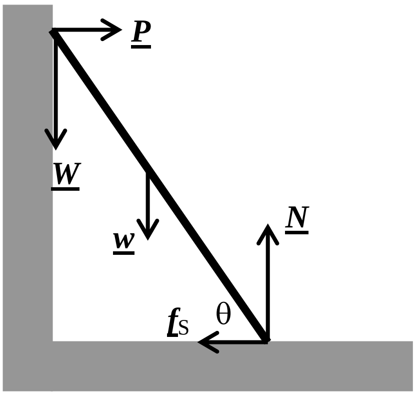
$$\theta = \tan^{-1}\left(\frac{1}{2\mu_S}\right).$$

Note that for vanishingly small coefficient of friction, $\mu_S \to 0$*, the only stable position for the ladder is* $\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$ *, so the ladder stands vertically against the wall!*

Question 3m

A firefighter has to climb to the top of the ladder in example 3.13. Suppose the ladder is of mass *m* and the firefighter is of mass *M*. Show that the shallowest angle for stability of the ladder is given by the equation

$$\theta = \tan^{-1}\left(\frac{1+m/2M}{\mu_S(1+m/M)}\right).$$



Solution: the new feature here is the weight of the firefighter W *at the top of the ladder, while the weight of the ladder* w *acts half-way along.*

Resolving horizontally we have the same equation as before:

$$P = f_S \qquad (i)$$

Resolving vertically we have

$$N = W + w = (M+m)g \qquad (ii)$$

Taking moments about the pint of contact with the floor, we find

$$(P)\times(L\sin\theta) = (W)\times(L\cos\theta) + (w)\times(L/2)\times\cos\theta = Lg\cos\theta\times(M+m/2)$$

This gives

$$P = \frac{(M+m/2)g}{\tan\theta}. \qquad (iii)$$

It follows that in the limiting case the shallowest angle of the ladder will be when

$$f_S = f_{S,\max} = \mu_S N$$

Now, substituting for N from (ii) above we get

$$f_{S,\max} = \mu_S N = \mu_S(M+m)g \ .$$

We now use (i) and equate $f_{S,max}$ to the value of P found in (iii):

$$f_{S,\max} = \mu_S (M+m)g = \frac{(M+m/2)g}{\tan\theta}.$$

This gives the required result:

$$\tan\theta = \frac{M+m/2}{\mu_S (M+m)},$$

dividing top and bottom by M gives

$$\theta = \tan^{-1}\left[\frac{1+m/2M}{\mu_S (1+m/M)}\right].$$

Let's put some numbers in. Suppose the firefighter plus equipment has a mass of 80kg, the ladder has a mass of 40kg and the coefficient of friction is 0.75. Then

$$\theta = \tan^{-1}\left[\frac{1+40/(2\times 80)}{0.75\times(1+40/80)}\right] = \tan^{-1}\left(\frac{1.25}{9/8}\right) = 48^\circ$$