

# Session 19

## Relativity

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# Welcome

Welcome to session 19.

In this session we'll look at both the special and general theories of relativity.

## Session Author

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Session Editor – Tim Puchtler

# Learning Objectives

After completing this session students should be able to:

- Define and use proper time
- Explain what is meant by The Minkowski metric and its relation to proper time
- Derive the effect of motion on Time measurements
- Explain how relativity accounts for the constancy of the Speed of light
- Define Relativistic energy
- Give an account of the Principle of relativity
- Distinguish between the special and general theories of relativity
- Describe the General theory of relativity as a theory of gravity by means of the Equivalence Principle -
- Describe the Schwarzschild metric and its application to Black holes
- And outline the idea of Wormholes for time travel.

# The Problem

The source below shows a secret memo from the files of MI6 which appears to condone the suppressing of information about the crash landing of an alien spacecraft in Roswell USA in 1947. Initial indications were of an ultra-advanced technology that reduced the time of travel from a planet in the Reticulum constellation 39 light years from the Earth to only 3 months.

*Area 51 Briefing Document*

*The following document has been obtained from MI6 files*

*TOP SECRET RESTRICTED CIRCULATION*

*Additional information on Area 51 (Roswell UFO)*

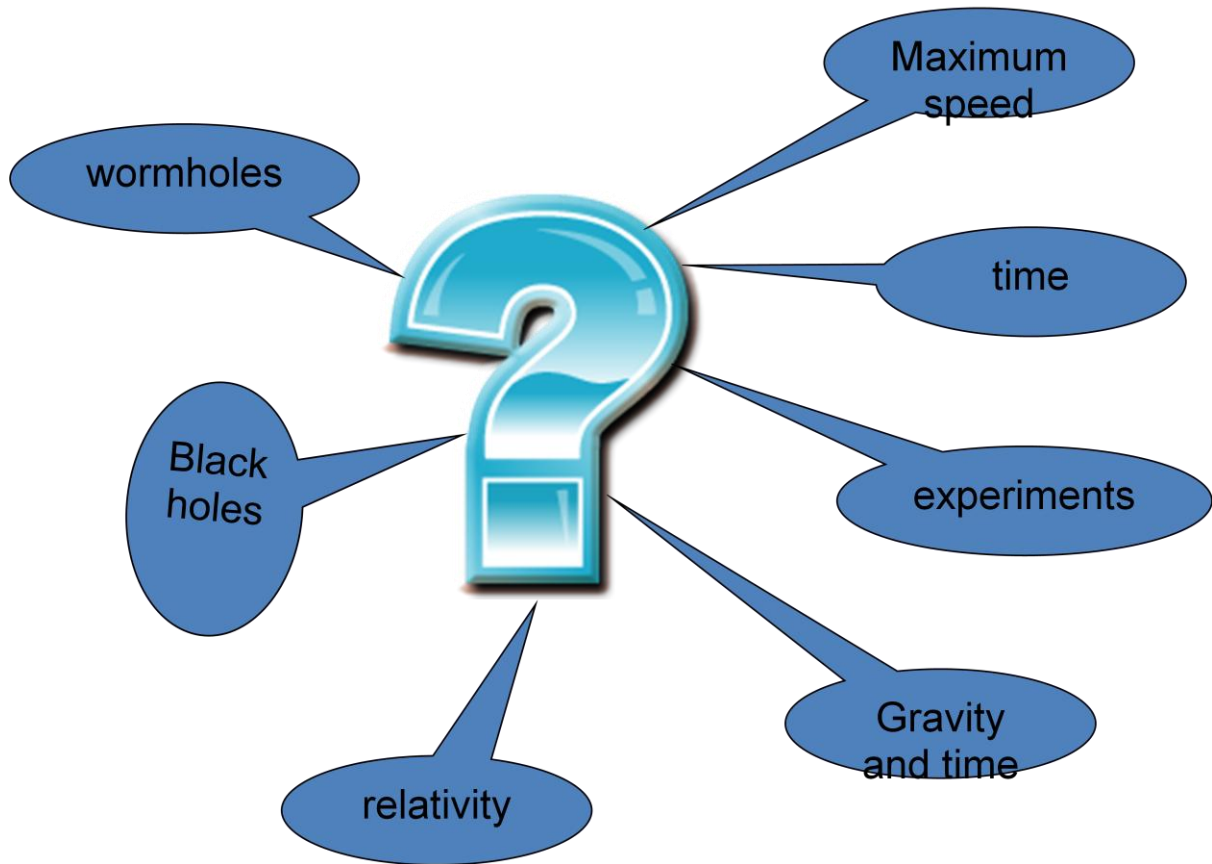
*[REDACTED] examination of partially charred material within the crash zone appears to indicate an expected journey time of three months [REDACTED] designation of the home world as zeta (2)-Reticuli [REDACTED] advanced wormhole technology [REDACTED]*

*Initial investigations concluded that this was part of a hoax to discredit the US military. The physical limitations on near light speed travel even allowing for the effects of special relativity on proper time were thought to rule out any nearby star for the alien host planet. However, recent advances in technology (especially GPS systems) have verified the importance of gravitational effects on time dilation. Exploratory studies in general relativity have failed to rule out the possibility of wormhole structures in spacetime that might be constructed by an advanced civilisation having access to negative energy 'matter' despite the paradoxes that appear to arise by allowing travel into the past.*

Can we provide the scientific case to support or demolish this argument?

# Learning Issues

## Learning Issues

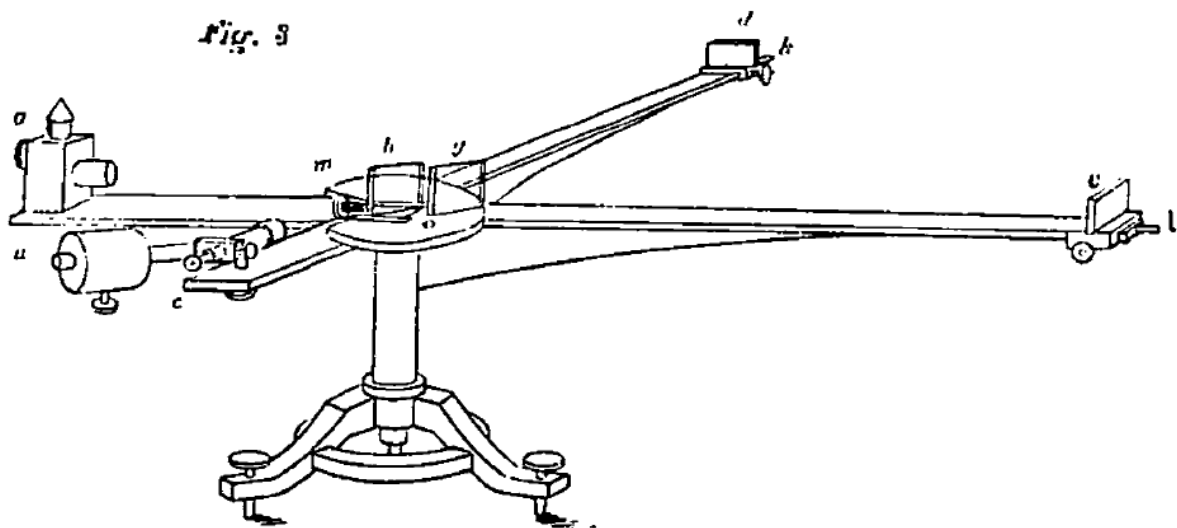


# Proper Time

## Speed of light

In the case of sound waves, the speed at which they move through space depends on the speed of the air through which they are moving. A fast flowing wind carries the sound towards us more rapidly than a still atmosphere. The speed of sound also depends on the speed of the observer and of the source, which gives rise to the Doppler effects for sound.

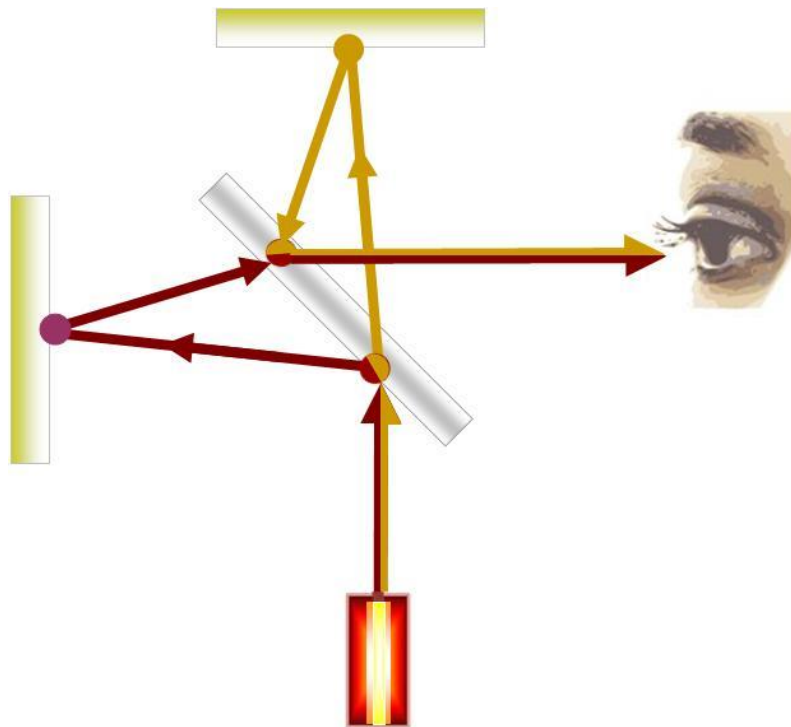
On the other hand the speed at which light reaches us is known to be independent of the medium, the source and the observer. No matter how you measure the speed of light, it has always the same value in a vacuum. This has been shown in numerous experiments starting from those of Michelson and Morley in 1887. A beam of light is split into two paths at right angles. When the two beams are recombined, interference fringes are produced at the detector. If the apparatus is rotated so that the two arms interchange their orientation, then because of the motion of the Earth, one would expect to see the fringe pattern move if light behaved like sound. No such fringe shift is observed.



Image<sup>1</sup>

<sup>1</sup> Michelson interferometer perspective illustration, Albert Abraham Michelson , as posted on commons.wikimedia.org. Public domain (licence expired).





Image<sup>2</sup>

How then is it possible for light always to have the same speed? As long as we can neglect the effects of gravity the answer is provided by the special theory of relativity; later we'll meet the general theory which includes gravity. According to special relativity the key to understanding the constancy of the speed of light lies in the notion of time.

"Where?" and "when?" Are different for different observers – the relativity of simultaneity.

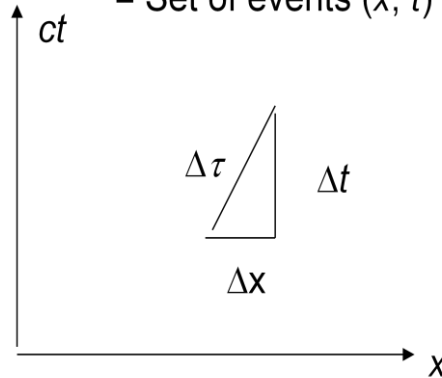
## Proper time

In special relativity we consider observers in uniform relative motion – exactly the same inertial observers we use in our discussion of Newtonian physics. A useful tool for picturing problems in relativity is the spacetime diagram. A spacetime diagram is drawn with the horizontal axis the x-axis of some inertial observer's coordinate system, and the vertical axis  $c$  times the time measurements of this observer.

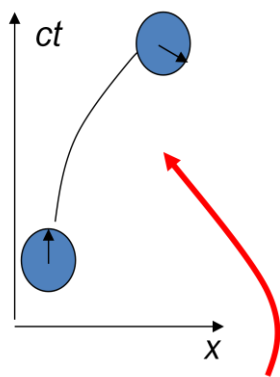
<sup>2</sup> Stylized Michelson interferometer, Brews ohare, as posted on [www.wikimedia.com](http://www.wikimedia.com). Creative Commons Licensed.

### Space-time diagram

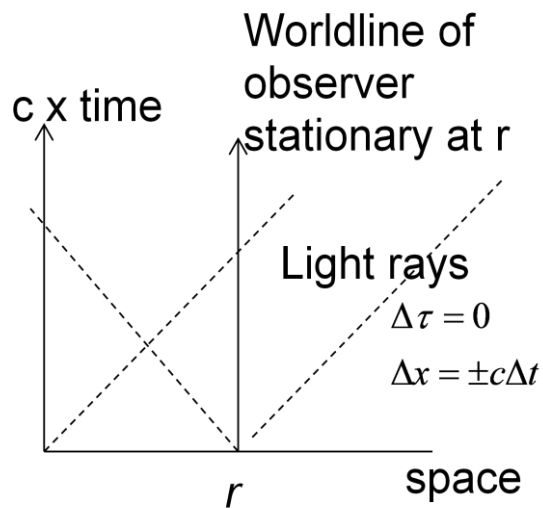
= Set of events  $(x, t)$



In this diagram each point represents an event – something happening instantaneously at a point – and each event is labelled by its coordinates  $(t, x)$ . More generally, we would use  $x, y$  and  $z$  to label a spatial location, so events would be labelled by coordinates  $(t, x, y, z)$ .



Proper time : time as measured on a co-moving clock



In special relativity it is possible to choose the  $x, y, z$  and  $t$  labels of two events, separated by  $\Delta x, \Delta y, \Delta z$  and  $\Delta t$ , such that time measured by a clock moving at constant speed between them, measures a time interval  $\Delta\tau$  given by equation (1) :

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \quad (1) \text{ (Minkowski Metric)}$$

$\Delta\tau$  is called the proper time interval between the event. Notice that we cannot talk about the proper time of an event, only about proper time differences.

This is a very subtle statement that captures virtually all of relativity. It says three things: first that there *are* such coordinates  $x$  and  $t$  in which the proper time takes this form. Such coordinates are called Minkowski coordinates and this expression for proper time is referred

to as the Minkowski metric. Second it *defines* proper time in these coordinates. In fact, thirdly, there are many such coordinates that could be used all of which give an expression of the same form and the same value for  $\Delta\tau$ . These different coordinates correspond to the measurements made by different observers in constant relative motion, the so-called inertial observers. Thus, two observers in relative motion will use different labels for events,  $(x, t)$  and  $(x', t')$  say, but they will agree on the proper time  $\Delta\tau$  between any two events. For the record, the relation between any acceptable pair of coordinates  $(x, t)$  and  $(x', t')$  is called a Lorentz transformation, but it is not necessary to go into that here.

## Principle of relativity

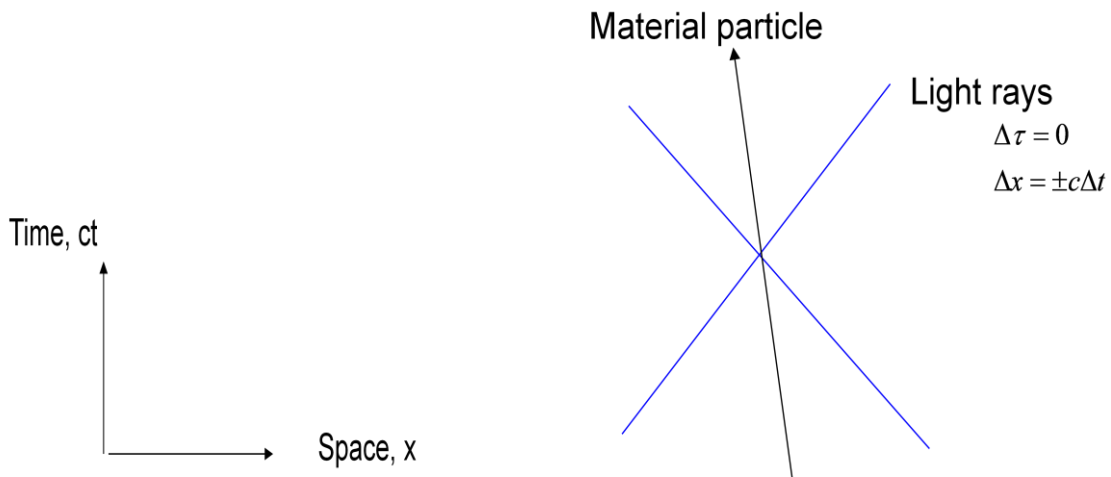
We can think of the measurement of a proper time difference between a pair of events as a simple experiment on the outcome of which observers in constant relative motion will agree. In fact they will agree on the outcome of any local experiment. This is the principle of relativity, which states that it is impossible for an observer to measure their absolute speed: only relative speeds have a meaning.

In particular we can think of measuring the speed of light. Suppose that an observer using  $x$  and  $t$  coordinates measures the speed of light to be  $c$ . Along this light ray we then know that  $\Delta x = c \Delta t$ , so the proper time lapse between any pair of events connected by a light ray is zero. But the proper time is the same for all observers, in particular for observers  $x'$  and  $t'$ . Thus  $c^2\Delta t'^2 - \Delta x'^2 = \Delta\tau^2 = 0$  and hence  $\Delta x' = \text{plus or minus } c \Delta t'$ . So the light ray also travels at speed  $c$  in  $x', t'$  coordinates.

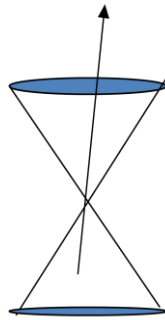
The speed of light is built into the structure of space-time; the speed of sound depends on material properties. The speed of light is an upper limit to the speed of a particle or signal; for  $v > c$ ,  $\Delta\tau$  is imaginary.

## The light-cone

If we restrict consideration to one space dimension, the light rays in spacetime are represented by lines  $x = \pm ct$  with a unit slope. With two spatial dimensions the light rays arriving at and leaving an event generate a cone, and with three spatial dimensions a hypercone, which we cannot visualise.



Since material particles must move at less than the speed of light they are represented by paths that lie within the cone.

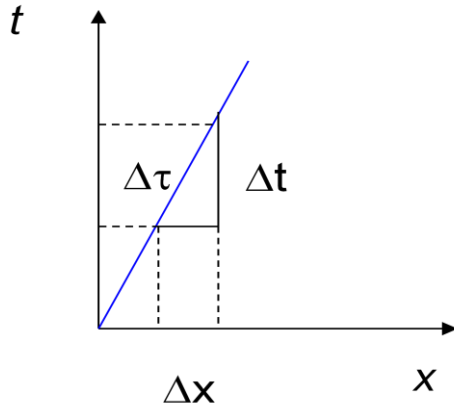


In higher dimensions the light surface is a "cone"

## Time dilation

Let's see what this all tells us about space and time.

The figure shows a spacetime diagram in which a stationary observer measures a coordinate time interval  $\Delta t$  using two synchronised clocks, one at each event.



The clock of the observer moving with speed  $v$  between those events records the proper time  $\Delta\tau$ .

$$\Delta x = v\Delta t \quad \text{As seen by the stationary observer}$$

We can use the Minkowski metric, (or equivalently the expression for proper time) to derive the relation between  $\Delta t$  and  $\Delta\tau$ . This we do by substituting  $\Delta x = v\Delta t$  into equation (1).

$$c^2\Delta\tau^2 = c^2\Delta t^2 - v^2\Delta t^2 \quad (1)$$

$$\Delta\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t$$

The result is equation (2)

$$\Delta t = \gamma\Delta\tau \quad (2) \text{ (Time Dilation)}$$

$$\text{Where } \gamma = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

Note that for non-zero  $v$  gamma is greater than 1. The time measured by the stationary observer is greater than the proper time of a clock moving between the events. Thus a moving clock runs slow relative to the synchronised stationary clocks of the observer. We refer to this as time dilation.

An observer at rest relative to the object measures the length  $\Delta l_0$  to be greater than the length  $\Delta l$  measured by an observer who is moving relative to the object. In fact,

$$\Delta l = \frac{\Delta l_0}{\gamma} \quad \text{(Length Contraction)}$$

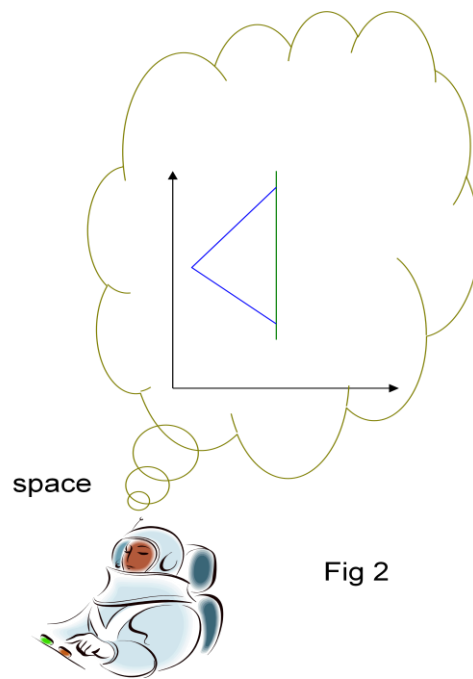
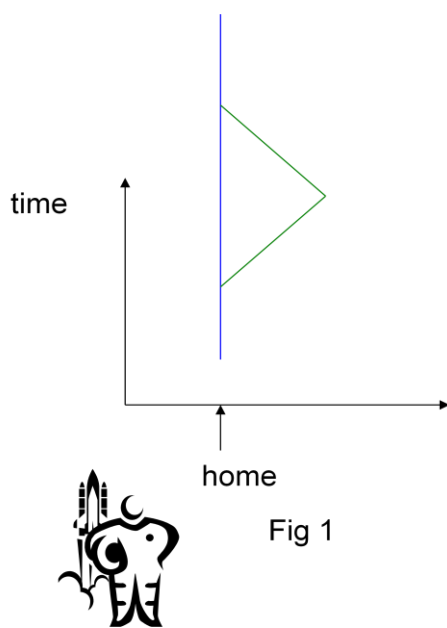
We refer to this as length contraction.

## Paradoxes

Some students find the notion that lengths and times are relative hard to accept, partly because they appear to lead to a number of paradoxes which are sometimes put forward as inconsistencies. We'll deal with two of these paradoxes here. Most of them can be resolved by constructing a spacetime diagram and being careful to avoid Newtonian concepts. In fact, it is important to understand that the special theory of relativity is logically consistent and does not lead to paradoxes. Whether it is the correct description of nature is decided by experimental evidence, which to date has found no deviation from the theory.

### 1. Twin Paradox

The famous twin paradox is illustrated here. The stay-at home twin is represented by the blue line. The green line represents the twin astronaut who travels out through space and back again. In the first figure the proper times measured along the green and blue lines are not the same. Time dilation means that the travelling astronaut returns home with less time elapsed than the stay-at-home twin. The paradox appears to arise if we consider the situation from the frame of reference of the travelling twin. Relative to him the Earth moves away and then comes back again. So his twin on the Earth should have measured a shorter elapsed time. They cannot both be right.

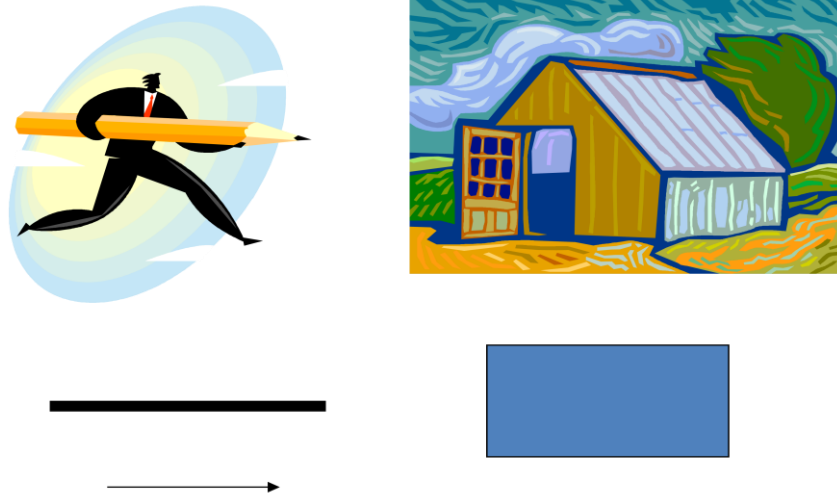


The resolution is that the two situations are not symmetrical. The astronaut going into space has to undergo an acceleration and deceleration that does not affect the stay-at-home twin. The time-difference on the constant velocity parts of the motion is real and the effect is measurable (as we shall see later).

A nice way to understand the twin paradox is to note that the twins follow different paths in spacetime, and that proper time measures the path length in spacetime. Just as we do not expect different paths in space to have the same length, so we do not expect different paths in spacetime to have the same proper time.

## 2. Barn Door Paradox

If a pole is too long to fit into the barn you can always try running at it. From the point of view of the intrigued on-looker this will contract the length of the pole and if you run fast enough the pole will fit in the barn and the onlooker can close the door once you're in. From the point of view of the runner, this is pointless: the approaching barn will be even shorter than the pole so it will be impossible for the onlooker to close the door. We therefore appear to have a paradox: either the runner can get the pole into the barn or he cannot.



Image<sup>3</sup>

In fact the pole fits in the barn. The problem from the point of view of the runner is that when the front end hits the wall and stops the back end goes on moving. In fact it cannot come to a halt until an elastic wave propagates down the pole to stop the motion – which must occur at a speed less than that of light. In both cases the pole tries to return to its length when at rest once the door is shut!

## Summary

- The speed of light has the same constant value in all frames of reference and is an upper limit to the speed of any object or signal
- Clocks measure proper time
- Proper time is given by  $c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$
- Along a light ray  $\Delta\tau = 0$
- Moving clocks run slowly by a factor  $\left(1 - \frac{v^2}{c^2}\right)^{1/2}$
- Moving rods are contracted by a factor  $\gamma$

<sup>3</sup> Images from Microsoft Clipart.



## SAQs

1. The star system zeta-Reticuli is 39 light years away. How long would it take to get here at a speed  $0.99c$  as measured by the space travellers?
  - (a) 39.39 years
  - (b) 3.9 years
  - (c) 0.39 years
  
2. A muon is an unstable elementary particle with a half-life of  $2 \times 10^{-6}$  s.
  - (i) What would be its range without time dilation if it is travelling at close to the speed of light? (Give your answer in metres to the nearest integer)
  - (ii) If the range is in fact 1200 m what is its speed?
    - (a)  $0.75 c$
    - (b)  $2c$
    - (c)  $0.87 c$
  
3. According to relativity events occur at definite proper times. Is this true or false?
  
4. The proper time between two events is independent of the path taken between them. True or false?

The answers appear on the following page

## Answers

1. (a) Incorrect: This is the time in Newtonian physics without time dilation, or the time as measured on Earth.  
(b) Correct: the proper time is a factor of gamma less than the Newtonian time  
(c) Incorrect: you appear to have used both time dilation and length contraction. You can use length contraction instead of proper time: but if you imagine the length of space between Earth and the star coming towards the space travellers at rest, there is no time dilation from their (non-) motion.
2. (i) 600 (accept 590-610) The range is just the lifetime times the speed of light (since we know that it is travelling close to the speed of light).  
(ii)  
(a) Incorrect: This is  $v^2/c^2$  not  $v/c$   
(b) Incorrect: This is gamma not the speed – the speed cannot exceed  $c$   
(c) Correct: This is the nearest to the correct answer. Gamma is obtained from  $c \times \text{gamma} \times \text{time} = \text{range}$ , so  $\text{gamma} = 2$  and then  $v/c$  is obtained from gamma. Strictly it should be  $v \times \text{proper time} = \text{range}$  but  $v$  and  $c$  are close and this way you avoid solving a quadratic equation. If you prefer you can iterate to get a more accurate answer:  $0.87 \times c \times \text{gamma} \times \text{time}$  gives a better estimate of the range
3. False: Only the difference in proper time between events is defined.
4. False: it is a useful approximation for small differences (and exact in the limit,  $\Delta x \rightarrow 0$ ,  $\Delta t \rightarrow 0$ ), but in general the proper time depends on the route taken.

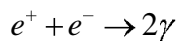
# Energy

$$E=mc^2$$

Perhaps the most iconic equation of physics is Einstein's  $E=mc^2$ . This tells us that if a system emits or absorbs energy  $E$  then its mass changes by  $E/c^2$ . This applies to any form of energy, so, for example, a hot kettle is more massive than a cold one; if a chemical reaction gives out heat, then the products are less massive than the reactants. These changes are too small to be noticeable in everyday life. Nuclear reactions provide more dramatic illustration of the amount of energy available from a relatively small change in mass.

This relation can be derived from the relativistic mechanics, but we shall not prove it here. Instead we'll check it in a couple of examples.

As a first example consider the annihilation of an electron-positron pair into two gamma rays:



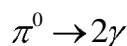
*Energy of two gamma ray photons = 1.022 MeV*

Let's take the mass of the electron in kilograms and see how much energy we expect:

$m_e = 9.1 \times 10^{-31}$  kg so  $E = mc^2$  gives an energy of  $2 \times 9.1 \times 10^{-31} \times (3 \times 10^8)^2$  Joules. To turn this into electron volts recall that an electron volt is the energy acquired by an electron in moving through a potential difference of 1 volt, that is the electron charge times 1 volt. So 1 electron volt is  $1.6 \times 10^{-19}$  Joules. Dividing by this gives an energy of 1.02 MeV: this is just what we find the energy of the gamma rays to be.

Our second example is Pion decay.

A pion is an unstable elementary particle that can decay into 2 photons.



The energy of the gamma rays can be measured as 135 MeV, so the mass of the pion should be  $135\text{MeV}/c^2$ , which is just what we find. So we have again verified  $E = mc^2$ .

Notice that the conventional way of quoting particle masses is as an energy divided by  $c^2$ . To turn this into more conventional units for a mass in kilograms we use the fact that 1 electron volt is  $1.6 \times 10^{-19}$  Joules. Dividing the energy by  $c^2$  gives the pion mass in kilograms.

In SI units  $M_\pi = 135 \text{ MeV}/c^2 = 135 \times 10^6 \times 1.6 \times 10^{-19}/(3 \times 10^8)^2 = 2.4 \times 10^{-28} \text{ kg}$

## Energy and momentum

$E=mc^2$  applies to a body at rest. There's an equally useful formula for bodies in motion. If a body has momentum  $p$  and mass  $m$  then the associated energy is given by equation (1).

$$E^2 = p^2 c^2 + m^2 c^4 \quad (1)$$

If  $p=0$  this reduces to  $E=mc^2$ . If  $m=0$ , for example for light, then  $E=pc$ . This gives us the relation between the energy and momentum for a photon. It also tells us that massless particles travel at the speed of light.

**Show that if  $p = \gamma mv$  then  $E = \gamma mc^2$**

**Show that  $E/p = c^2/v$  and hence that  $v=c$  for a massless particle**

**We can interpret this as an increase in mass as a result of motion**

For a massive particle relativity changes the relation between velocity and momentum. Instead of the Newtonian expression for momentum,  $p = mv$ , we have  $p = \gamma mv$ . Using this in equation (1) we can get an alternative expression for the energy of a moving mass, namely  $E = \gamma mc^2$ . We'll leave this as a problem. These equations for energy and momentum can be interpreted by saying that the mass of a body in motion is increased from  $m$  to  $\gamma m$ . The kinetic energy is the total energy minus the rest mass energy:  $\gamma mc^2 - mc^2$ .

*The kinetic energy is:  $\gamma mc^2 - mc^2$*

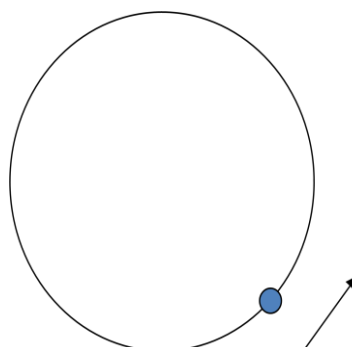
Finally, let's debunk the myth that  $E=mc^2$  led to the atomic bomb: The relatively large release of energy associated with the fission of nuclei was discovered experimentally quite independently of this relation. The mass differences must be consistent with these energy releases, but the atomic bomb could have been developed without the theory of relativity.



Image<sup>4</sup>

## Experiments

There are a large number of experimental tests of relativity. One of the most dramatic comes from engineering: it is the enormous size of particle accelerators.



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<sup>4</sup> Operation Crossroads Baker, United States Department of Defense, as posted on commons.wikimedia.org. Image available as Public Domain.

Let's start by working out the radius of a synchrotron accelerator according to Newtonian physics. Equation (1) is the equation of motion of a particle of charge  $e$  moving with a circular speed  $v$  in a magnetic field  $B$ .

$$Bev = m \frac{v^2}{r} \quad (1) \quad (F=ma \text{ from Newtons laws})$$

The left hand side is the force and the right hand side the mass times acceleration for motion in a circle of radius  $r$ . In terms of the momentum  $p = mv$  this simplifies to equation (2):

$$p = Ber \quad (2)$$

We want the radius in terms of the energy of the particle,  $E$ , so we use  $E = \frac{1}{2} mv^2$  in the form  $E = p^2/2m$  to get (3)

$$E = \frac{p^2}{2m} = \frac{B^2 e^2 r^2}{2m} \quad (3)$$

With some rearrangement the result (4) for the Newtonian radius  $r_N$ .

$$r_N = \frac{(2mE)^{1/2}}{Be} \quad (4)$$

Now let's do the same calculation in relativity: The mass of the particle will increase as a result of its motion, so the first change is that in equation (1)  $m$  is replaced by  $\gamma m$ . The second change is the relation between energy and momentum: for sufficiently high speeds the contribution of the rest mass to the total energy is small, so we neglect it and approximate the relation between energy and momentum by equation (5). The result for the radius  $r_E$  of the relativistic synchrotron is equation (6).

$$Bev = \gamma m \frac{v^2}{r} \quad (4) \quad E = cp \quad (5) \quad r_E = \frac{E}{eBc} \quad (6)$$

We can now work out the ratio  $r_E/r_N$ .

$$\text{Ratio: } \frac{E/eBc}{(2mE)^{1/2}/Be} = \left( \frac{E}{2mc^2} \right)^{1/2} = \left( \frac{\gamma}{2} \right)^{1/2} \gg 1$$

It turns out to depend on the ratio of the total energy to the rest mass energy of the particle, a ratio which is enormous for modern high energy synchrotrons. Clearly a synchrotron designed using Newtonian mechanics would not work.

The LHC has a radius of 8 km and reaches  $\gamma$  factors of  $7.5 \times 10^3$ . What would be the radius of the equivalent Newtonian synchrotron?

## Summary

- If a system emits or absorbs energy  $E$  then its mass changes by  $E/c^2$
- Energy  $E$  and momentum  $p$  are related by  $E^2 = p^2 c^2 + m^2 c^4$
- Equivalently,  $E = \gamma mc^2$ , so  $\gamma = \frac{E}{mc^2}$

## SAQs

- How much more does a 2 litre kettle of boiling water weigh than one at 0°C
  - 0 kg
  - $9 \times 10^{-12}$  kg
  - $2 \times 10^{-15}$  kg
- If a reasonable proper time for the journey from zeta-Reticuli is 3 months what  $\gamma$  factor is required? Give your answer to the nearest integer.
  - How much energy is required to achieve this for a  $10^5$  kg spacecraft (the mass of the Shuttle)? Give your answer in Joules to the nearest integer power of 10.
- A muon is an unstable elementary particle with a half-life of  $2 \times 10^{-6}$  s. If the mean energy of a muon at creation is 6 GeV what is its range? The muon mass is  $106 \text{ MeV}/c^2$ 
  - 600 m
  - 34 000 km
  - 34 km
- What is the pressure on a silvered solar sail subject to a flux of radiation  $F \text{ W m}^{-2}$ .
  - $F \text{ N m}^{-2}$
  - $F/c \text{ N m}^{-2}$
  - $F/c^2 \text{ N m}^{-2}$
  - $2F/c \text{ N m}^{-2}$

The answers appear on the following page



## Answers

- (a) Incorrect: a change in energy is always associated with a change in mass

(b) Correct: the heat input is  $4.18 \times 10^3 \times 2 \times 100 \text{ J}$  and this has to be divided by  $c^2$

(c) Incorrect: you may have omitted the specific heat of water.
- (i) 156. Distance =  $\gamma \times \text{proper time} \times c$  (since the speed will be very close to that of light), so  $\gamma = 39 \text{ light years} / 0.25 \text{ years} = 156$

(ii) 24. The kinetic energy is  $(\gamma - 1) Mc^2$  (the minus 1 comes from the fact that we need only the energy in excess of the rest mass);  $\gamma$  is  $39/0.25=156$  from the relation for proper time, given that the speed of the spacecraft is close to that of light. So  $KE = 155 \times 10^5(\text{kg}) \times c^2 = 10^{24} \text{ J}$  (about 1000 times the annual world total)
- (a) Incorrect: You have forgotten time dilation

(b) Incorrect: you have probably taken giga to mean  $10^{12}$  not  $10^9$

(c) Correct:  $\gamma$  is the ratio of energy to rest mass energy, around 60; the lifetime is enhanced by a factor of  $\gamma$  and the speed is very close to  $c$ .
- (a) Incorrect:  $F$  is the energy flux not an energy density

(b) Incorrect: Using rate of change of momentum = energy flux/ $c$  gives this answer, but it's wrong by a factor of 2 because the photons are reflected by the solar sail

(c) Incorrect: The relation between energy and momentum is  $p = E/c$ .

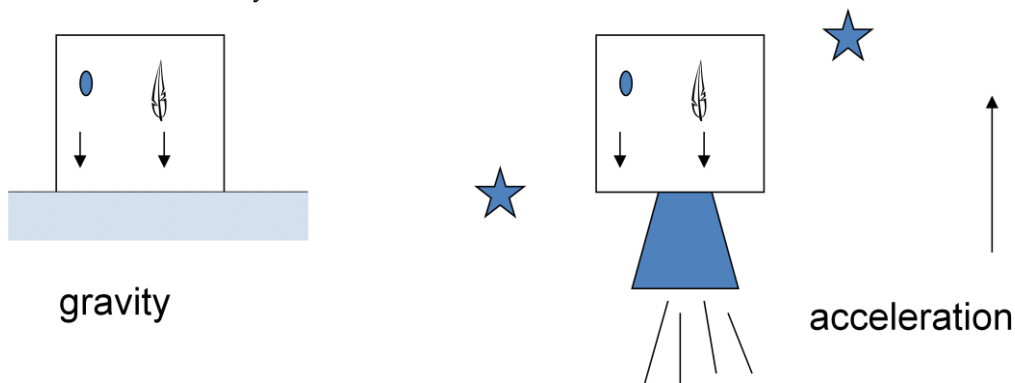
(d) Correct: the rate of change of momentum =  $2 \times \text{energy flux}/c$  where the factor 2 comes from the fact that the photons are reflected

# Gravity

The huge energy requirements make it unlikely that special relativistic effects would account for the short journey time from zeta Reticulum, so we now turn to gravity and the general theory of relativity.

## Equivalence Principle

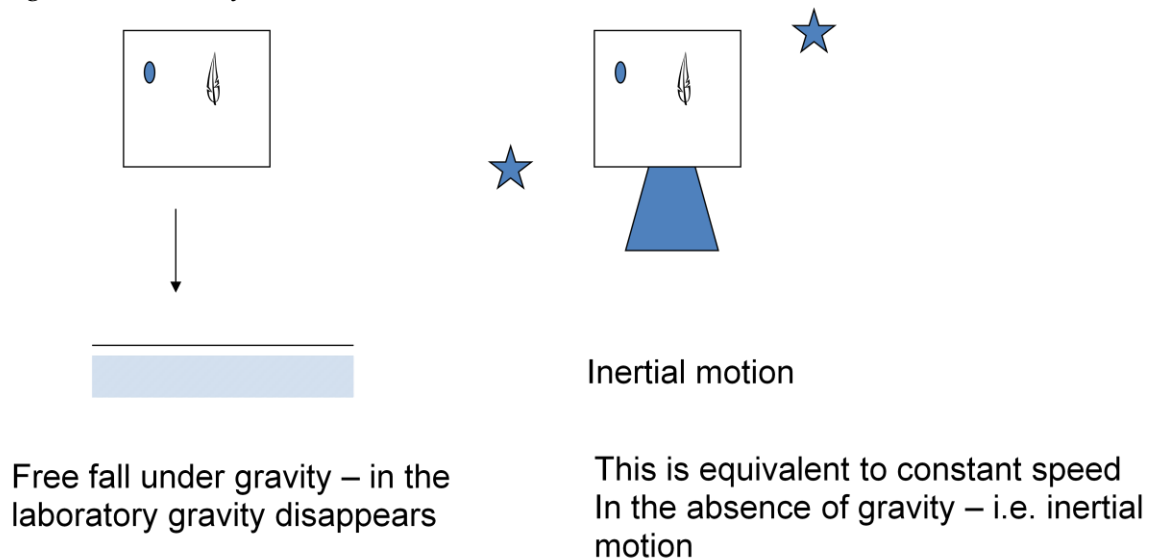
General relativity is often held to be extremely difficult: it's true that the detailed treatment requires an extensive mathematical background. But the basic idea, like all great ideas, can be expressed quite simply. The basic physical principle underlying relativity is what is known as the equivalence principle. We'll gloss over some of the minor intricacies and present the idea informally.



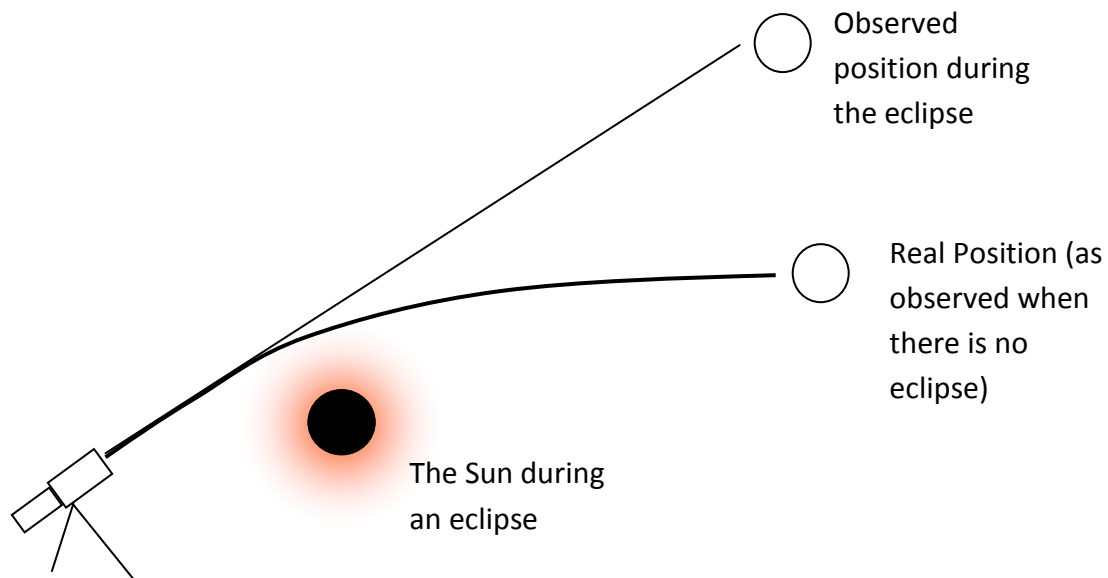
Above see an experiment performed under two apparently different circumstances. In the one case a feather and a weight are dropped without air resistance on the surface of a planet where the acceleration due to gravity is  $g$ . In the other case the feather and the weight are dropped in a rocket that is subject to a constant acceleration  $g$  in gravity-free space. From the outside we can tell the difference. From the inside we can't because all bodies fall with equal acceleration in a gravitational field. In other words, in this experiment acceleration and gravity are equivalent in the sense that they have equivalent effects that cannot be distinguished. The principle of equivalence extends this to all local experiments – where local includes the fact that we can't get information from outside the laboratory – no local experiment can distinguish between the effects of gravity and acceleration. Note that this applies only to gravity: subject to other forces the feather and the weight would not fall with equal acceleration so the force could not be replaced by any acceleration.

## Equivalence Principle (2)

A second version of the equivalence principle is as follows. Imagine you are in free fall towards an airless planet: then gravity disappears as everything in your laboratory and the laboratory itself would fall at the same rate. Someone in the laboratory would not be able to tell whether they were in free fall or in an inertial laboratory in deep space. In this latter laboratory, or reference frame, we know what the laws of physics are: they are precisely the laws as they apply without gravity. To make gravity appear again we transform back to a non-freely falling frame – one with an acceleration of  $g$  -- and this transformation gives us the laws of physics in the presence of gravity. This transformation is what the mathematics of general relativity achieves.



Starting from the principle of equivalence Einstein was able to formulate his general theory of relativity. The general theory predicts that light rays are bent in a gravitational field and determines the orbits of the planets more accurately than Newton's theory. It also predicts the existence of gravitational waves, black holes and possibly wormholes.



## Schwarzschild Metric

To understand how gravity is described by spacetime curvature let's take a specific example: let's see how a mass like the Earth affects the space around it. By producing a "curvature of spacetime" we mean that the presence of a massive body affects the proper time between any two events. For a spherical non-rotating body the new expression for proper time is called the Schwarzschild metric:

$$c^2 \Delta\tau^2 = c^2 \left(1 - \frac{2GM}{rc^2}\right) \Delta t^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \Delta r^2 - r^2 \Delta\theta^2 - r^2 \sin^2 \theta \Delta\phi^2$$

*(Schwarzschild Metric)*

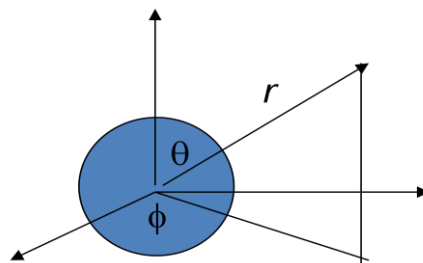
You can see how this differs from the Minkowski metric in the absence of gravity by having coefficients that depend on the radial coordinate distance  $r$  from the massive body. Setting  $M = 0$ , the Schwarzschild metric reduces to the Minkowski metric in polar coordinates.

An appropriate analogy is the way in which the curvature of the Earth affects flight times between airports. You cannot construct a map of a flat Earth from an idealised airline timetable. This analogy is why the expression for proper time is said to correspond to a curved spacetime.



Image<sup>5</sup>

A similar analogy is with the lines of longitude on the Earth's surface. Pairs of points at different latitudes, separated by the same longitude, are not the same distance apart: this is an effect of curvature. Similarly, in spacetime, equality of coordinate separation does not imply equality of proper time separation.



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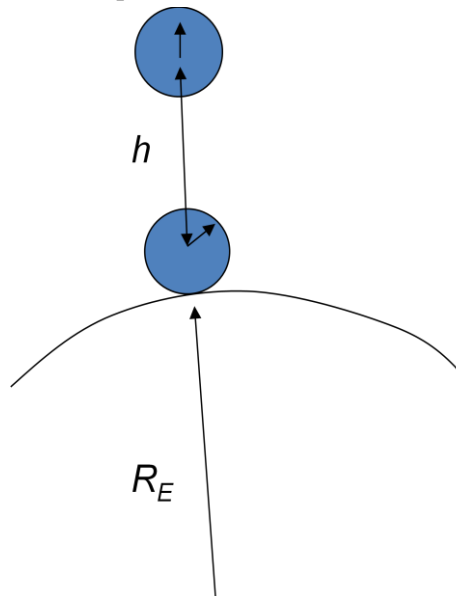
<sup>5</sup> Fitz globe, Norman B. Leventhal Map Center at the BPL, as posted on [www.flickr.com](http://www.flickr.com). Creative Commons Licensed.

In the equatorial plane,  $\theta = \pi/2$ , the metric becomes:

$$c^2 \Delta\tau^2 = c^2 \left(1 - \frac{2GM}{rc^2}\right) \Delta t^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \Delta r^2 - r^2 \Delta\phi^2$$

## Stationary clocks

Our first result is that clocks run slower in stronger gravitational fields. To see this, consider two clocks at fixed locations, one in space and one on the Earth's surface.



Because their spatial coordinates are fixed any two readings of the clocks are events at the same location. The proper times between such events, separated by coordinate time  $\Delta t$ , can be read off from the Schwarzschild metric as equations (1) and (2).

$$\begin{aligned} \Delta\tau_S &= \Delta t \left(1 - \frac{2GM_E}{(R_E + h)c^2}\right)^{1/2} & (1) \\ &= \Delta t \left(1 - \frac{2GM_E}{R_E \left(1 + \frac{h}{R_E}\right) c^2}\right)^{1/2} \\ \Delta\tau_E &= \Delta t \left(1 - \frac{2GM_E}{R_E c^2}\right)^{1/2} & (2) \end{aligned}$$

$(1 \pm x)^p \cong 1 \pm px$   
 for  $x \ll 1$

The mass and radius of the Earth are denoted by  $M_E$  and  $R_E$ . The result is the ratio of proper times in equation (3).

$$\frac{\Delta\tau_E}{\Delta\tau_S} \approx 1 - \frac{GM_E}{R_E c^2} \frac{h}{R_E} \quad (3) \quad (\text{Ratio of times})$$

The algebra to get to this simplified approximate result uses the binomial expansion several times: this tells us that if  $x \ll 1$  then  $(1 \pm x)^p$  is approximately equal to  $1 \pm p x$ . Here both  $h/R_E$  and  $2GM_E/R_E c^2$  are much less than one. Note that at infinite distance from a gravitating body proper time and coordinate time are equal: this means that clocks at infinity measure coordinate time  $t$ .

**At infinite distance from a gravitating body  $\Delta\tau = \Delta t$  so clocks at infinity measure  $t$**

Show that  $2GM/c^2$  for the Earth is about 1 cm and hence that  $\frac{2GM_E}{R_E c^2} \ll 1$

See also: <http://math.ucr.edu/home/baez/physics/Relativity/SR/experiments.html> for papers on experimental tests

## Red shift

Let's see what else we can discover from the Schwarzschild metric. The next effect we find is called the gravitational red shift. Imagine light rays emitted from the surface of a star of radius  $R_s$  to the Earth at a distance  $r_E$ . To make things simple we'll neglect the gravitational field of the Earth. Because the situation is static – nothing is changing in the metric – the  $\Delta t$  between successive wave crests at the star is the same as the  $\Delta t$  between those crests at the Earth.

But the time we measure is not  $\Delta t$  – it's the proper time  $\Delta\tau$ , and this is different on the star and on the Earth, because gravity affects the rate at which clocks run. We've given the details of the calculation below: you should be able to follow the first step in which we relate  $\Delta\tau$  to  $\Delta t$  using the Schwarzschild metric around a star of mass  $M$ . Since we are considering fixed points in space at  $r$  and  $R_s$  we have  $\Delta r = 0$ . The algebra that follows uses the binomial expansion to simplify the result.

$$(1-x)^{1/2} \cong 1 - \frac{1}{2}x$$

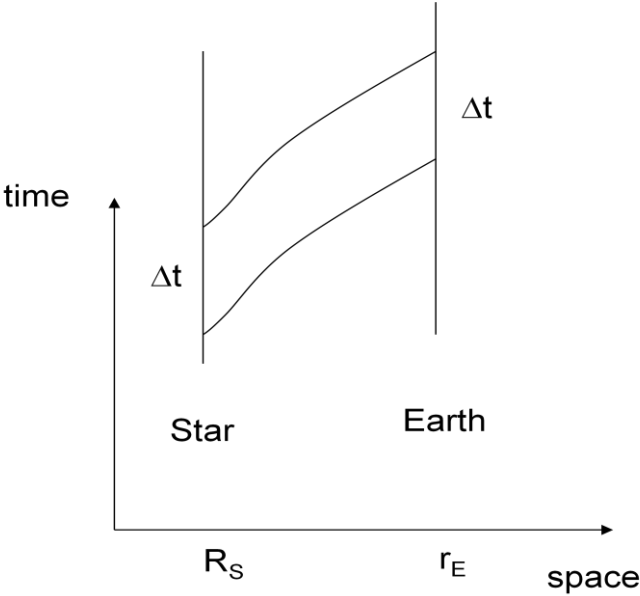
$$\frac{\Delta\tau_E}{\Delta\tau_S} = \frac{\left(1 - \frac{2GM}{r_E c^2}\right)^{1/2} \Delta t}{\left(1 - \frac{2GM}{R_S c^2}\right)^{1/2} \Delta t}$$

$$\approx 1 + \frac{GM}{c^2} \left( \frac{1}{R_S} - \frac{1}{r_E} \right)$$

$$\approx 1 + \frac{GM}{R_S c^2} \quad (r_E \gg R_S)$$

$$\frac{\lambda_E}{\lambda_S} = \frac{c\Delta\tau_E}{c\Delta\tau_S} = 1 + \frac{GM}{R_S c^2}$$

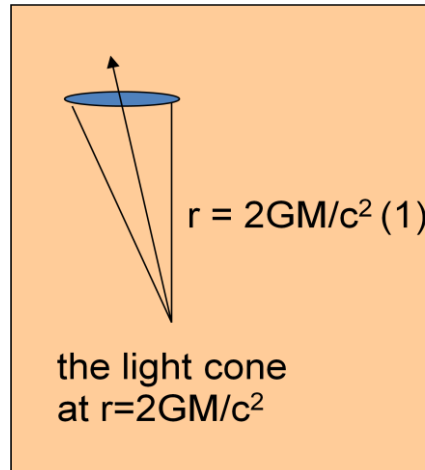
Of course, we don't measure proper times between wavecrests – we measure wavelengths or frequencies. The observed effect is therefore a shift in wavelength,  $\lambda$ , of a spectral line, and since the proper time is expanded, the wavelength is greater: that is we observe a red shift.





## Black holes

Suppose now that the star has shrunk within  $r=2GM/c^2$  and that a light ray is emitted radially outwards from  $r = 2GM/c^2$ .



Then  $\Delta\tau = 0$  for any pair of events on the ray, and  $\Delta\theta = 0$  and  $\Delta\phi = 0$  for such events since the ray is radial. For the metric therefore we have equation (2).

$$0 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2GM}{rc^2}\right)^{1/2}} \quad (2)$$

From this it follows that:

$$\frac{\Delta r}{\Delta t} = \left(1 - \frac{2GM}{rc^2}\right) c = 0$$

at  $r=2GM/c^2$ .

So the radial coordinate of the light ray never changes: it remains fixed at  $r=2GM/c^2$  and the light is never received by an external observer. So  $r=2GM/c^2$  is a one way surface, or event horizon. The radius  $r=2GM/c^2$  is called the Schwarzschild radius and the object is called a black hole.

## Light cones near a Schwarzschild black hole

The section shows spacetime diagrams in the vicinity of a black hole using different coordinates from those we used to introduce the Schwarzschild metric. Although the coordinates are no longer  $t$  and  $r$  all measurable quantities are unchanged. The new coordinates have been chosen to make the behaviour of light near a black hole clearer. The light cones are the surface  $\Delta\tau = 0$ ; notice how they tip over as the surface  $r=2GM/c^2$  is approached so that material particles are forced to move inwards. This is the effect of gravity on light. The two diagrams show the same effect in slightly different ways. Notice in figure 2 how the light rays from the infalling trajectory diverge in time as they move outwards. This illustrates the increasing red shift as the infalling body approaches  $r=2GM/c^2$ . The one-way light-like surface  $r=2GM/c^2$  is an Event Horizon because we cannot see inside it. In the next section we'll see why we cannot use black holes to travel in time.

Figure 1:

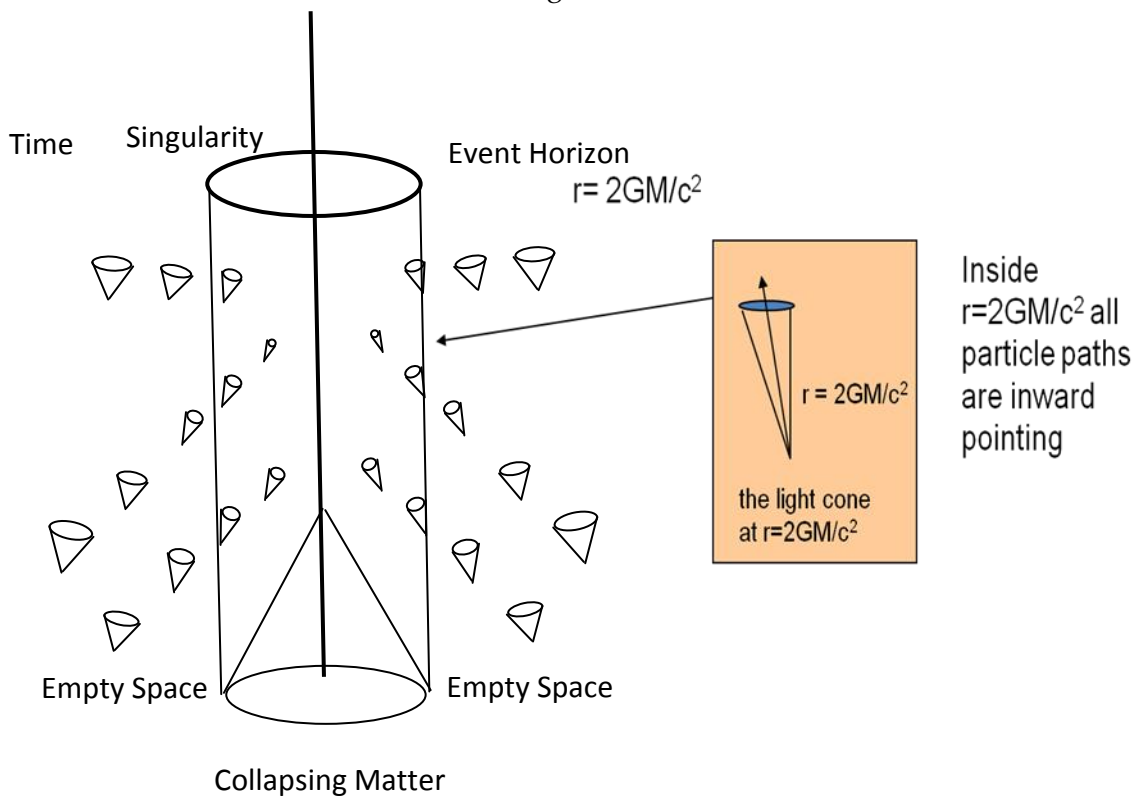
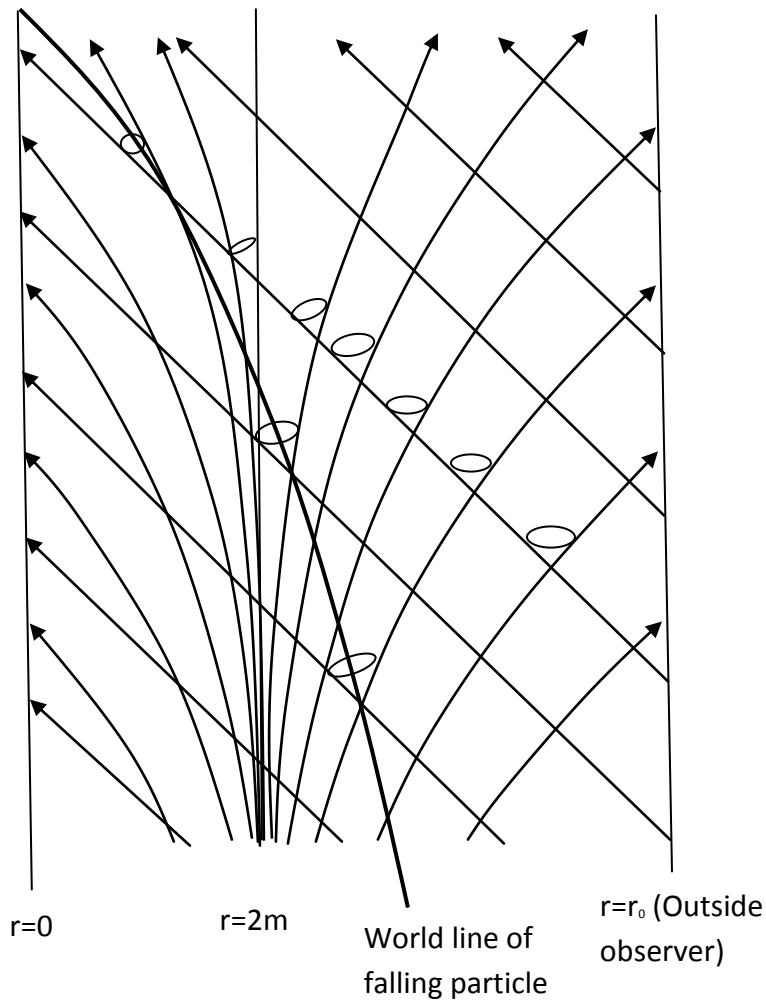


Figure 2:



## Examples of Black Holes

The existence of black holes, which is a predication of relativity not found in Newtonian physics, appears to be well-established in various situations. There is evidence for black holes a few times the mass of the Sun in binary star systems, a black hole of about 3 million solar masses at the Centre of the Milky Way and supermassive black holes of more than 1000 million solar masses at the centres of other Galaxies.

You can work out the density of material in a black hole from the mass and the Schwarzschild radius for that mass.

Galactic Centre:



Image<sup>6</sup>

Invisible black hole companion in a binary star system:



Image<sup>7</sup>

Also, a supermassive black hole lies at the centre of this galaxy.

**Work out the mass densities of these black holes**

## Summary

- The equivalence principle implies that in free fall the laws of physics are those of special relativity

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<sup>6</sup> Chandra X-Ray Observatory Image of Sagittarius A. Images property of NASA.

<sup>7</sup> M33x7, as posted on [http://commons.wikimedia.org/wiki/File:M33x7\\_scale.jpg](http://commons.wikimedia.org/wiki/File:M33x7_scale.jpg). Creative Commons Licensed.

- In the presence of a gravitating body the metric takes on a non-Minkowskian form
- The metric exterior to a static spherical body is the Schwarzschild metric
- Clocks run slow under gravity
- Light is redshifted by gravity
- The Schwarzschild radius  $r = 2GM/c^2$  is an event horizon or black hole

## SAQs

Answer (a) True or (b) False?

1. Black holes can only be formed from matter at high density
2. An observer crossing the event horizon will be stretched radially
3. It takes an infinite time to fall into the singularity
4. To what size would the Sun have to shrink for it to become a black hole? (The mass of the Sun is  $2 \times 10^{30}$  kg.)  
Give your answer to the nearest integer number of km.

The answers appear on the following page

## Answers

1. False: The density of a body is proportional to  $M/R^3$ ; if  $R$  is proportional to  $M$ , as in a black hole, then the density is proportional to  $1/M^2$  or  $1/R^2$ , hence goes down as  $M$  and  $R$  go up. A black hole can be formed at any density provided the mass and radius are large enough.
2. False: the observer notices nothing special on crossing the horizon, although for a small enough hole and a large enough body the normal variation of gravity with height would be to stretch the body. This however is not specifically a property of black holes.
3. False: the diagram shows light rays falling into the singularity in a finite time. An observer would follow a path interior to the light cones hitting the singularity at  $r=0$  in a finite proper time
4. 3. The Sun would have to shrink within the event horizon at  $r = 2GM/c^2 = 3 \text{ km}$ .

# Time Travel

Here we'll address the issue of time travel and try to resolve the Roswell incident.

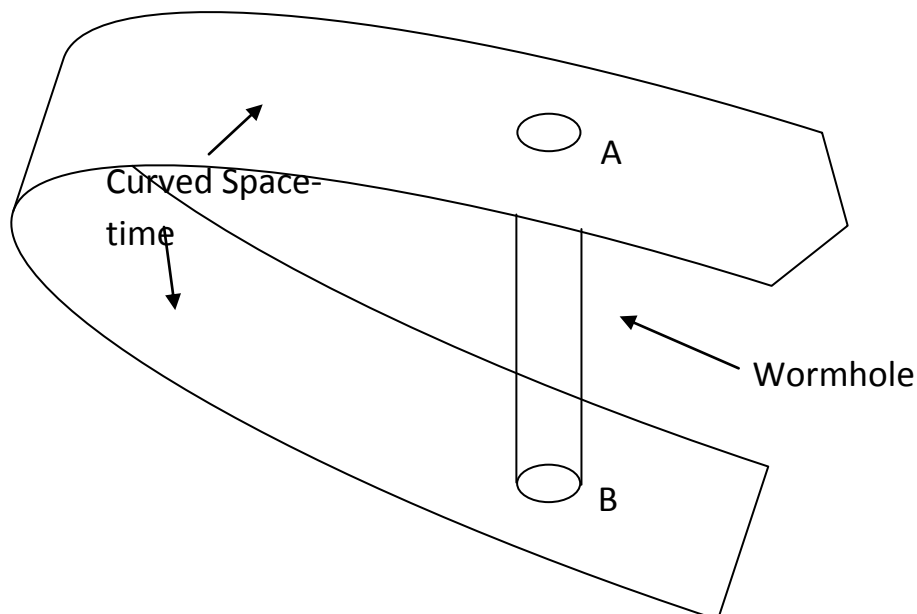
## Time Travel Agenda

These are the issues we'll look at.

1. Black holes aren't time machines
2. Wormholes
3. Making a wormhole with negative energy
4. Wormhole time travel
5. Problems of time travel

## Wormhole travel

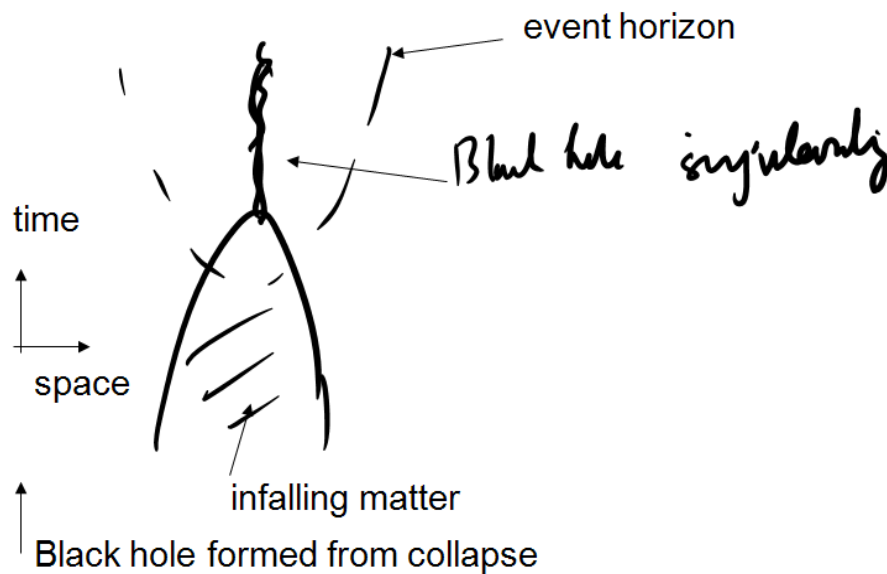
Here we show how a wormhole might be used as a shortcut across space. The proper time of a body falling through the wormhole can be much less than that of a body moving between the same points in the exterior space.



Can we make a wormhole?

## Eternal Black Holes

If a black hole is formed from collapse then it contains a region of infinite density at its centre. We call such a region a “singularity”. Any infalling bodies must hit the singularity at  $r=0$ . In principle however, black holes could exist from the beginning of the universe constructed out of the vacuum of spacetime. These would be so called eternal black holes. In this case the picture looks different: there is another region of spacetime which emerges from the hole. There is nothing in the mathematics that says that this apparently different region cannot be our part of spacetime labelled differently. An analogy might be the different pages of a map where the overlap regions are not different instances of the same road layout but in fact are to be identified as the same roads on different pages.



Unfortunately this doesn't help us navigate our spacetime. It is impossible to travel across the bridge between the spacetimes, because there is a singularity in the way. Attempting to do so just lands you up in the singularity.

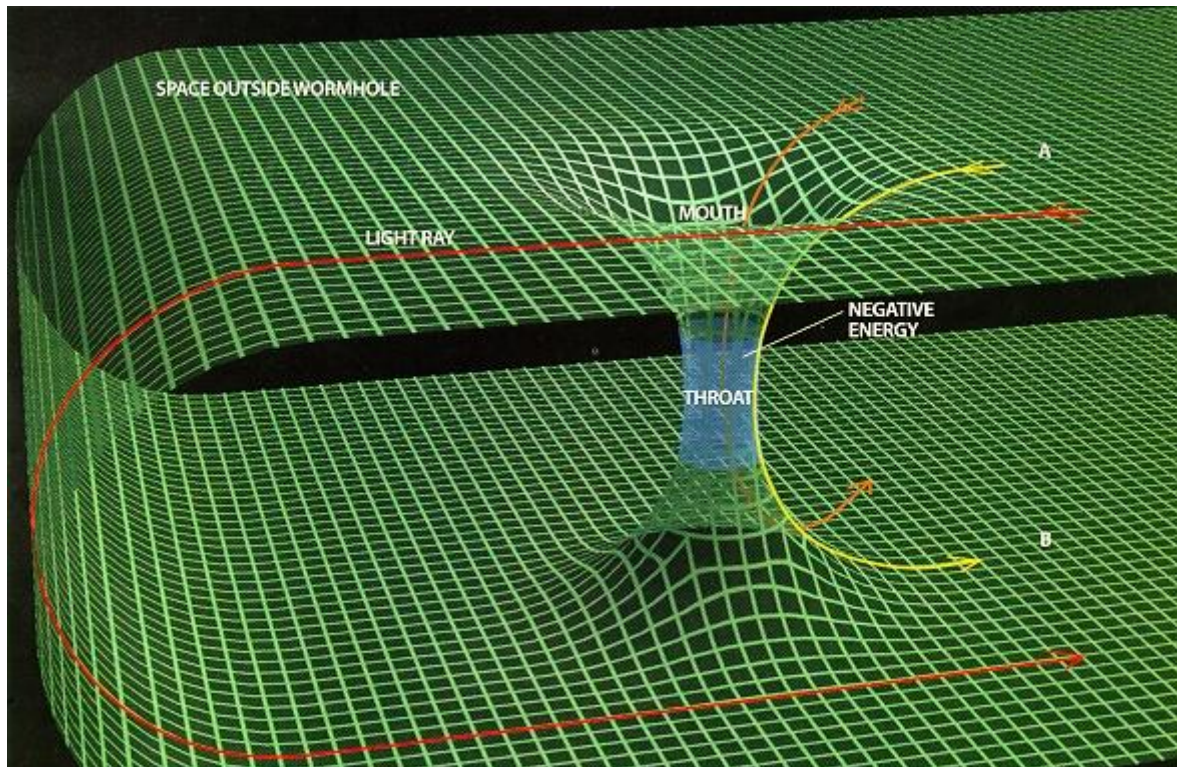
So this type of black hole cannot provide a shortcut, or wormhole, across space. In fact the only other types of black hole possible are rotating ones and electrically charged ones which also cannot be used to make wormholes.

A section at time  $t=0$  through the spacetime of an eternal black hole that avoids the singularity. Unfortunately the throat 'pinches off' more quickly than one can travel through it.



## Wormholes

Is there any other way to make a wormhole? The answer is that it cannot be made out of the vacuum of spacetime as pure gravity, nor can it be done with normal matter.



Image<sup>8</sup>

According to the theory of relativity we can make a wormhole only if we can use material which has negative energy relative to the vacuum. This is not at all the same as antimatter, which has positive energy – when matter and antimatter annihilate the energy does not cancel to zero – gamma rays are produced. But small amounts of negative energy can be created. The best known example is the region between two capacitor plates. We can understand this from the Uncertainty Principle which tells us that the plates must contain fluctuating currents. The resulting attraction between the plates is equivalent to a reduction in energy between the plates relative to the vacuum outside. Hence, the region between the plates contains negative energy. But this is a minute amount compared to what would be required for a macroscopic wormhole.

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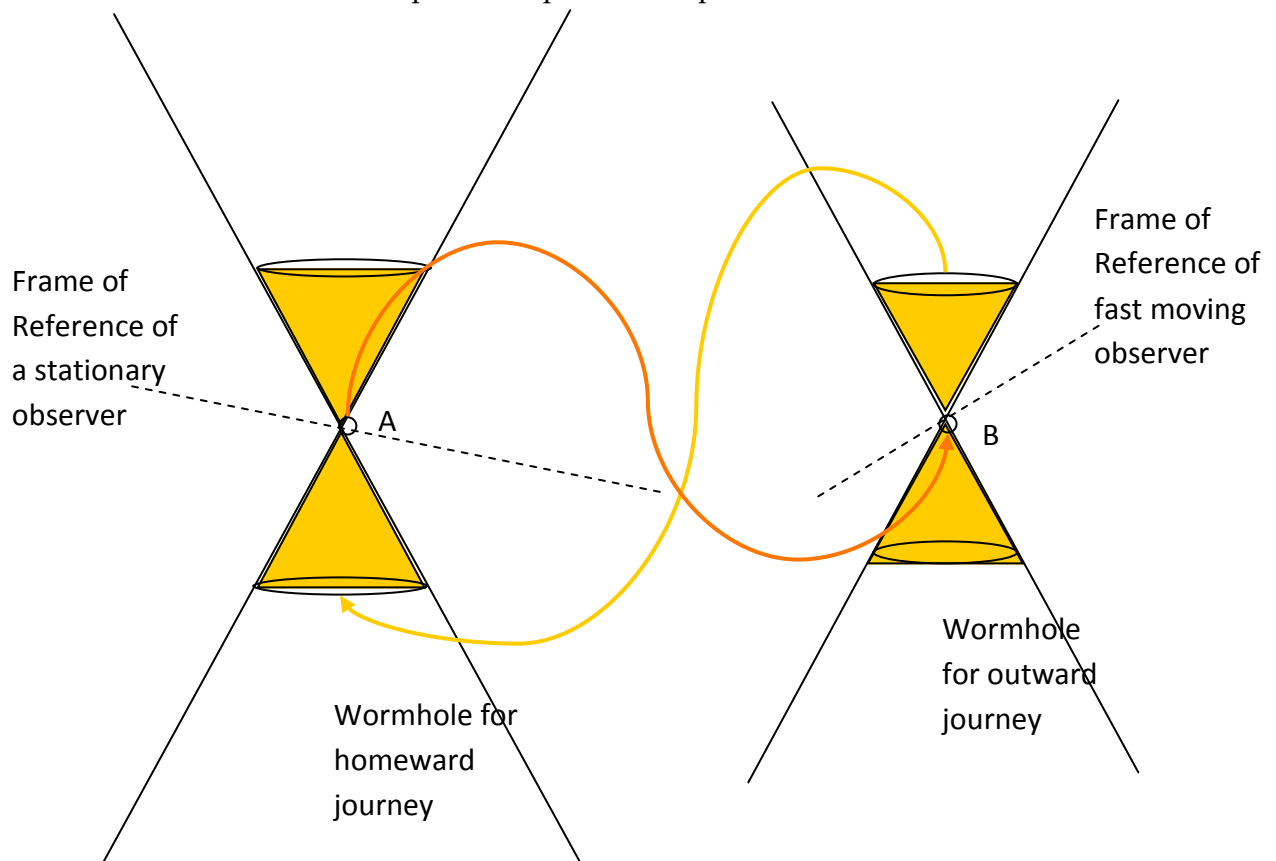
<sup>8</sup> A Wormhole, Benji64, as posted on commons.wikimedia.org. Creative Commons Licensed.

## Time machines

Let's however continue and see what an advanced civilisation, that has found how to make copious amounts of negative energy, might do with a wormhole. First, they could use it to create a shortcut as we've seen, by towing the wormhole, say by gravitational attraction, to the required destination.

However, they could also use it to create a time machine which would enable observers from the future to travel back to the time the wormhole was created (but no earlier). To do this start with ends A and B close together. Then accelerate end B, say, to close to the speed of light on a round trip so that a negligible few hours passes while at the other end A many years pass. If the observer at end A now moves through the wormhole they can come out at end B in the same location, but many years in their past.

The illustration also shows a trip into the past from alpha centauri to the Earth.



If one end of a wormhole has been accelerated close to light speeds time at that end will have progressed less than at the other end so a returning traveller will appear in the past

There are obvious problems with travel into the past – the most obvious is that one can arrange for oneself not to have been born, which is clearly a contradiction. Does this mean that wormholes are impossible? Unfortunately there appears to be nothing in principle in current physics which makes them so, although there are arguments which suggest that they might be unstable, hence not useable in practice to travel anywhere.

## Summary

- Black holes cannot be used for time travel
- Wormholes can provide shortcuts in spacetime, but require exotic matter to be traversable
- Traversable wormholes could be used (in future) for travel into the past thereby raising logical problems
- We'll leave you to draw your own conclusions about the aliens at Roswell.

## SAQs

Answer True or False:

1. Wormholes will allow us to travel into the future
2. Wormholes allow travel arbitrarily far back into the past
3. Negative energy matter is composed of antiparticles

The answers appear on the following page

## Answers

1. True: in the example given previously, instead of the observer at end A going to end B in the past, the observer at end B goes to end A in the future (where they meet their older companion who can tell him what has happened in the many years in the future)
2. False: the exit of the wormhole cannot be at an earlier time than it was first made
3. False: antiparticles have positive energy which is released when they annihilate with matter.

## Overall Summary

- Clocks measure proper time
- $E=mc^2$
- According to the equivalence principle gravity and acceleration are locally indistinguishable
- In the absence of gravity proper time is given by the Minkowski metric
- In the spacetime external to a static spherical mass, proper time is given by the Schwarzschild metric
- The speed of light is constant in all freely falling frames of reference
- The Schwarzschild metric also describes the spacetime of a black hole

## Meta tags

Author: Ted Thomas.

Owner: University of Leicester

Title: Enhancing Physics Knowledge for Teaching – Relativity

Keywords: Time; Energy; Gravity; sfsoer; ukoer

Description: In this section we'll look at both the special and general theories of relativity.

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Language: English

Version: 1.0



## Additional Information

This pack is the Version 1.0 release of the module. Additional information can be obtained by contacting the Centre for Interdisciplinary Science at the University of Leicester.

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