Session 16

Properties of Matter

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Contents

Welcome

Welcome to session 16

In this session we'll look at some of the properties of matter.

This session will be arranged differently to the other sessions, in that we will not have a single problem which we are focusing on, but rather introduce a new problem for each section.

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Learning Objectives

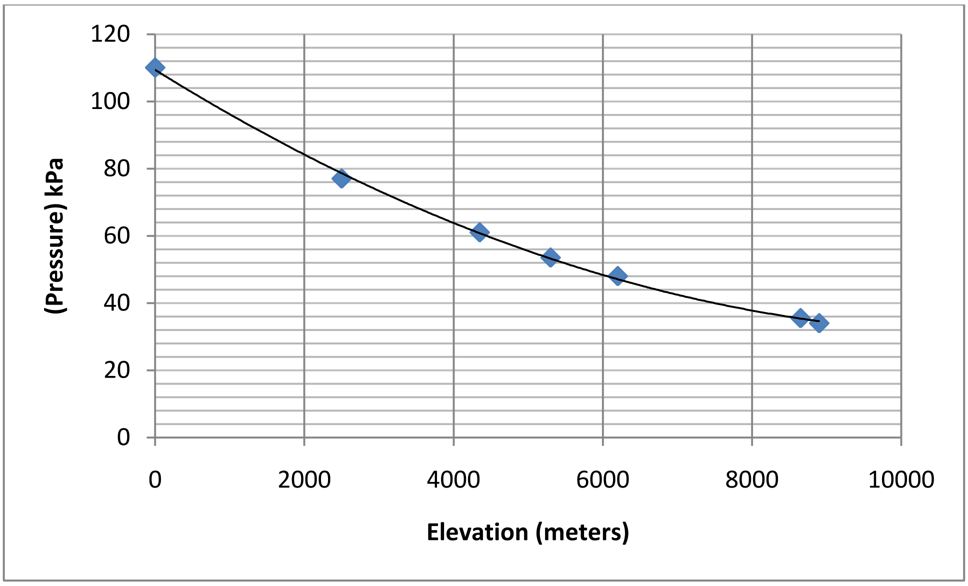
- Derive and use the Barometer law for fluids and gases
- State and apply Hooke's law
- Distinguish between stress and strain
- Show a knowledge of how the force between atoms can be used to derive bulk properties of crystalline solids
- Define surface tension
- State and apply Archimedes' principle
- Apply the continuity equation and Bernoulli's equation to fluid flow

Barometer Law

The Problem

Why is sport difficult at high altitude?

You probably know that the air gets thinner at higher altitude. The graph shows how the pressure of the atmosphere falls off as we go to the tops of higher and higher mountains. The problem is to understand why this is and how it affects exercising.



The weight of the atmosphere

Let's start by thinking of all the air above us: it has a weight which must press down on the Earth – or, equally, on the top of your head. To work out how much mass there is above each unit area, and hence what the force exerted by the atmosphere is, we can do a rough calculation.

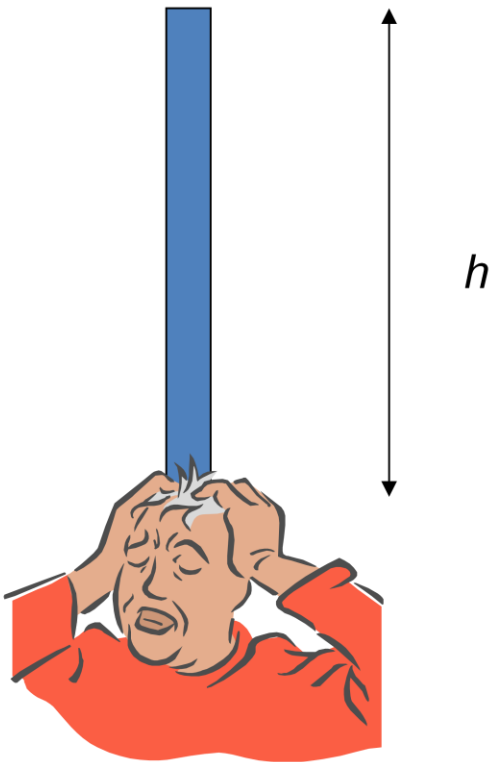


Image (Microsoft Clipart)

We need to know the height of the atmosphere, and its density. Of course there's no exact value to the height – the atmosphere just gradually thins out as we go higher. But we can take a height when say the pressure is half that on the ground and call it the height for the purposes of this estimate. The density of the atmosphere isn't constant – it too falls off with height, but for the purposes of an estimate we can assume it's approximately constant over this range of pressure. Then the weight of atmosphere per unit area is ρgh.

Weight of a unit area column of air = $\rho g h$

ρ = 1.21 kg m^{-3}

g = 9.81 m s^{-2}

h = 8.50 km

W = 101 000 N m^{-2}

This comes to over 100 metric tonnes weight per square metre.

What approximately is the value of atmospheric pressure at ground level in N m^{-2}? Explain.

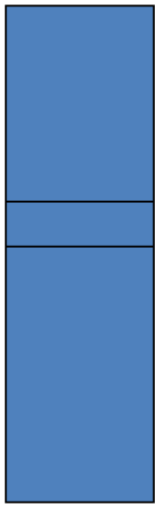
Why does pressure decrease with height?

The pressure decreases because as we ascend there is less weight of atmosphere above us to support.

We can use this to make a mathematical model of the change in pressure. If we ascend by a small amount dz the reduction in weight of a column of atmosphere of unit area is the density times the volume times the acceleration due to gravity, or ρgdz. This is just the reduction in pressure, giving us equation 1, where the minus sign indicates the decreasing pressure with increasing height.

$$(1)\quad dP = -\rho g dz$$

$$(2)\quad \rho = \frac{mP}{kT}$$



This relation contains two unknowns: both density and pressure. So to get any further we have to relate the density to the pressure. We can assume that air is an ideal gas so we can use the ideal gas law from session 14. Equation (2) states the law in terms of the density ρ and the molecular mass m. Of course, this equation introduces another unknown, namely the temperature. It's too complicated to include here how the temperature of the atmosphere is determined, and for our purpose it's unnecessary. Instead we make a simple model by assuming that the temperature is constant.

Think for a moment why it is reasonable to assume a constant temperature but not a constant density?

We also assume that we can treat air as if it were composed of 'air' molecules of mass m. In fact, as long as the composition of the atmosphere can be taken to be constant with height, we can just use the average molecular mass for m of about 29 times the mass of the hydrogen atom. This average comes from taking the composition of air to be 78 per cent nitrogen and 21 per cent oxygen.

With the assumption of constant temperature our equation for the run of pressure with height looks like equation (3) here:

$$\frac{dP}{dz} = -\frac{mg}{kT}P \qquad (3)$$

Instead of trying to solve this equation mathematically, we compare it with something we already know. Let us write λ for mg/kT so equation (3) becomes dP/dz = -λP; this is analogous to the radioactive decay law: dN/dt = -λN. As a result of radioactive decay, the decaying atoms decrease in number exponentially with a mean lifetime of 1/λ.

By comparison, we say:

Radioactive: $N = N_0 e^{-\lambda t}$

Pressure: $P = P_0 e^{-\frac{mg}{kT}z}$ (4)

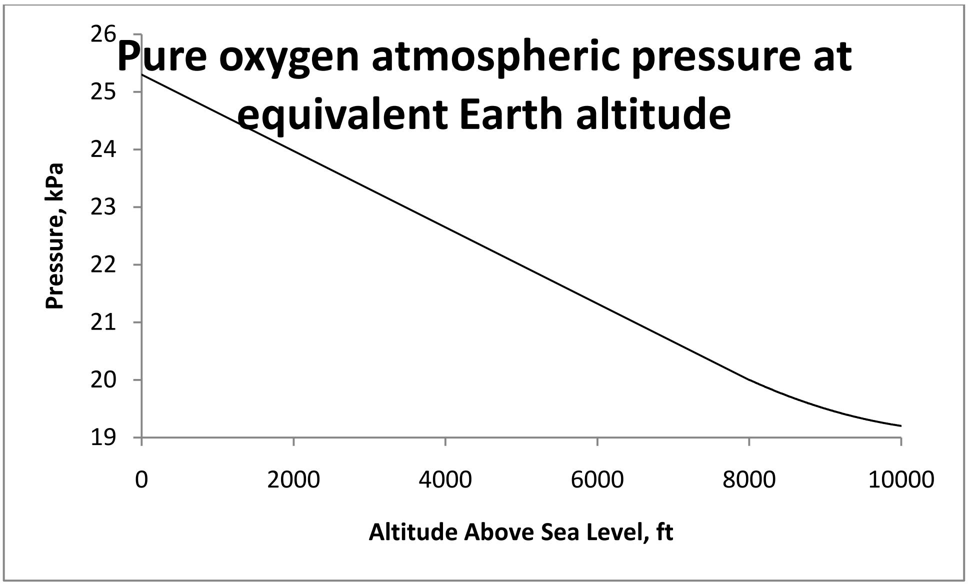
So the pressure in our model of the atmosphere must decrease exponentially with a mean height of one over the corresponding λ. We call 1/λ, which is equal to kT/mg, the "scale height" of the atmosphere. It is the height over which the pressure changes by a factor 1/e, in other words the height below which most of the atmosphere is situated.

The exponential law in equation (4) is called the Barometer formula.

Suppose that the atmosphere were to have the same sea–level values of pressure and density, but with a density that did not vary with height. Show that its height would equal the scale height of the real atmosphere. Hint: rewrite the scale height equation as *h = P*/(ρg)

Pressure of Oxygen with height

We are now in a position to solve our original problem. If the pressure in the atmosphere goes down, so does the partial pressure of the oxygen content. The graph shows the exponential decline in oxygen, which, to a good approximation, follows the overall decline in pressure:



This is obviously a problem for events where athletes need a supply of oxygen. For some events, where aerobic respiration is not important, there may be advantages to competing at altitude. Can you explain why the Mexico Olympics set a new and long-standing long jump record?

Summary

- The pressure in the atmosphere equals the weight of air above unit area
- The pressure falls off with height approximately in accordance with the Barometer Law: $P = P_0 e^{-\frac{mg}{kT}z}$
- This enables us to estimate the "scale height" of the atmosphere

SAQs

1. The acceleration due to gravity on the surface of Mars is 0.4g. Compared to the Earth how does this (alone) affect the scale height of the atmosphere?

   The scale height (a) increases (b) decreases (c) stays the same

2. The mean temperature on the surface of Mars is -63^{o}C. Compared to the Earth how does this (alone) affect the scale height of the atmosphere?

   The scale height (a) increases (b) decreases (c) stays the same

3. By considering which of the effects in questions 1 and 2 is the larger decide whether the scale height of the Martian atmosphere is larger or smaller than that of the Earth:

   (a) Smaller (b) Larger

4. The pressure of the Earth's atmosphere decreases with height because

   (a) Gravity gets weaker (b) molecules escape into space (c) the weight of air to be supported decreases.

The answers appear on the following page

Answers

1. (a) Correct: the formula gives h inversely proportional to g so as g goes down h goes up. This is clear physically because a weaker gravity exerts less pull.

   (b) incorrect: weaker gravity exerts less pull as can be seen from the formula for scale height.

   (c) incorrect: weaker gravity exerts less pull as can be seen from the formula for scale height.

2. (a) incorrect: an atmosphere will expand on warming as can be seen from the formula

   (b) correct: the formula gives scale height proportional to temperature; physically you can think of the atmosphere expanding on warming, so the scale height on Earth must be larger.

   (c) incorrect: an atmosphere will expand on warming as can be seen from the formula

3. (a) incorrect: the temperature change is about 83/293 ~ 0.3 (using the absolute scale) not 63/20 so this is less than the change in gravity.

   (b) correct: gravity changes by a factor 0.4, the temperature by a factor of about 83/293 ~ 0.3 so the change in gravity is the bigger effect.

4. (a) Incorrect: The scale height is about 0.02 of the Earth's radius, so gravity is effectively constant (as we assumed in the derivation)

   (b) Incorrect: molecules do escape but this does not affect the scale height, as can be seen from the formula.

   (c) Correct: as shown previously.

Elasticity

The Problem

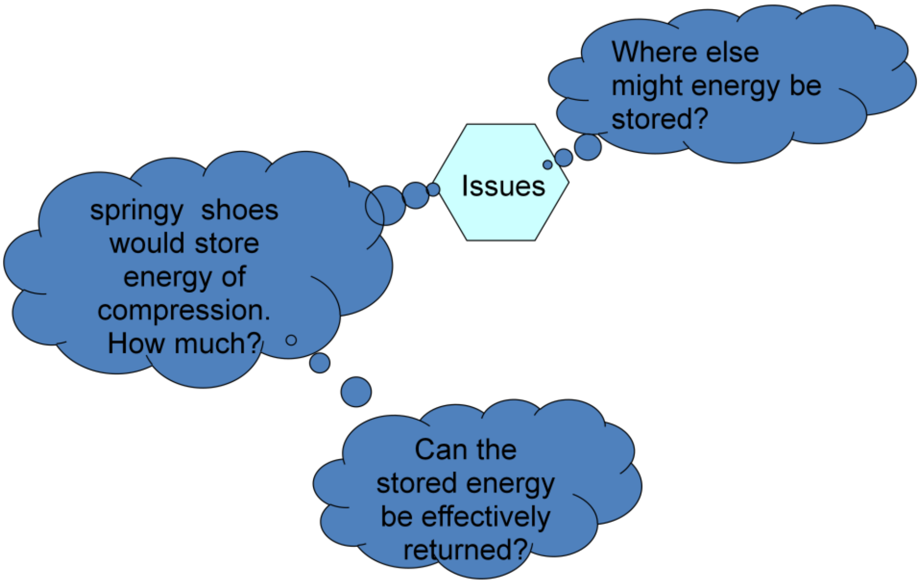
Why do sprinters generally not wear springy running shoes?

One might think that runners would get an advantage by wearing springy running shoes which would compress on contact with the ground and then spring back to help with the next stride. Is this potential advantage significant?

Image[1]

The picture shows the controversial prosthetic legs of Oscar Pistorius – how much of an advantage (if any) does he get from the energy stored between strides in his carbon fibre legs? We'll try to answer this question by the end of this section.

[1] Oscar Pistorius, by Elvar Freyr, as posted on www.flickr.com. Creative Commons Licenced



Stress

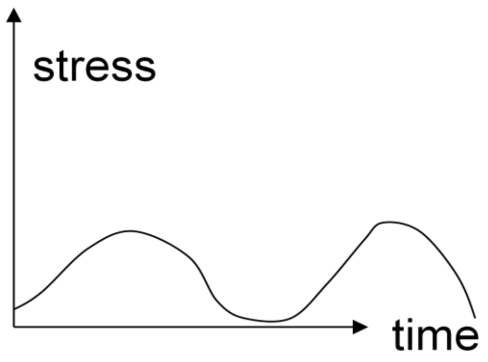
Energy is stored in the body wherever forces lead to elastic extension or compression – when the load is removed an elastic material will spring back releasing the stored energy.

The Achilles tendon stretches and relaxes during a stride. This is a significant contributor to the energy balance so it will give an estimate of the total amount of energy stored in each stride. So we need to investigate how to work out the energy stored when an elastic medium is stretched.

First we give a definition of the terms we shall use – stress and strain. It's important to use these correctly and not interchange them or use them imprecisely.

Let's start with stress: In physics this is defined as the force per unit area in a body. The forces on the Achilles tendon during running have been measured and the area of the tendon is known, so we can work out the stress.

Stress = Force per unit area



stress versus time in running

Match the graph of stress versus time to the motion of the foot during running. (What do the peaks and troughs correspond to?)

Stress in the Achilles tendon

This section gives some typical values. For a:

force of 4700 Newtons and a

cross section area of 89mm^2 the

stress is 53 x 10^6 Newtons per square metre or 53 mega Pascals.

The Pascal is the unit of stress in the SI system. It's sometimes useful to remember that one mega Pascal is also 1 Newton per square millimetre.

Stress has the same dimensions as pressure and is measured in N m^{-2} or *Pascals* (Pa) 1 MPa = 1 N mm^{-2}

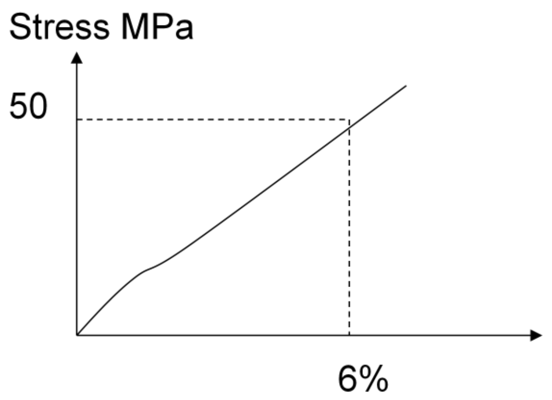
We also consider the following data that we'll need next: the length of the Achilles tendon is typically 250mm and the maximum extension is 15mm.

Strain

Next we define strain. In physics this has the precise meaning of fractional extension: that is if a body of natural length l is extended by an amount δl the strain is δl / l. Since this is a length divided by a length it is dimensionless.

Strain ε = fractional extension = extension/original length

For small extensions, δl/l < 0.1 say, many bodies satisfy a linear relationship between stress and strain: this is known as Hooke's law. The coefficient of proportionality in this relationship is called Young's modulus. Clearly, since strain is dimensionless, Young's modulus has the same dimensions as stress, that is force per unit area. So in the SI system, Young's modulus is given in Pascals (or Newtons per square metre).



Stress – Strain relation (Hooke's Law)

$$\sigma = Y\varepsilon$$

Strain

Y = Young's modulus

The graph shows the stress-strain relationship for the Achilles tendon – the slight departure from a straight line is real but, to a good approximation, it follows Hooke's law. From the graph we can read off the strain at a stress of 50 mega Pascals: it's .06 or 6%.

Stored energy

Now we are in a position to return to our problem and start to work out the stored energy in the tendon. The stress in the tendon at each stage depends on the extension. So the force, which is just the stress times the area, varies as the tendon is extended. Therefore to get the work done we have to integrate the force times dx where x runs from zero to the full extension. If the natural length is L the extension is just the strain ε times L. This written as follows:

$$\int F dx = \int \sigma A dx = \int AY \frac{x}{L} dx = \frac{ALY}{2}\varepsilon^2 = \frac{AL\sigma^2}{2Y} = \frac{1}{2}AL\sigma\varepsilon$$

$$= \frac{1}{2}(89\,\text{mm}^2)(250\,\text{mm})(53\,\text{N mm}^{-1})(0.06) = 35\,000 \text{ N mm} = 35 \text{ Nm} = 35 \text{ J}$$

There are various ways of stating the result in terms of the stress, or the strain or a combination of the two as in the final formula.

Substituting the values used previously we find that the energy stored in a tendon is 35 Joules. The total stored energy is around 100 Joules, relatively small compared to the energy of a runner:

Compare the KE of a marathon runner at 4.5 m s^{-1}:

$$\frac{1}{2}Mv^2 = \frac{1}{2} \times 70 \times (4.5)^2 = 708\text{J}$$

Comparison with running shoes

We're now in a position to make a comparison with the energy stored in running shoes. You might like to have a go with your own estimates before reading on.

We can easily estimate the stress from the weight of a body and the area of a running shoe. We've taken a 70kg runner and an area of 6000mm^2 so the stress is 700 N / 6000 mm^2 or about 0.12 N mm^{-2}. We can guess a value for the strain, but we should check that our guess is compatible with a reasonable value for Young's modulus. Rubber can undergo large strains and still obey Hooke's law, so let's say a running shoe of depth 30mm can be compressed by 10mm by a 70kg runner. The strain is 1/3, so Young's modulus has to be 0.12 / (1/3) or about 0.36 N mm^{-2}.

$$\frac{Mg}{A} = \frac{700\text{N}}{6000\text{mm}^2} = 0.12 \text{ N mm}^{-2}$$

This is on the high side for rubber although some plastics will come close. Let's see what it gives. Substituting our values in the formula we find that the stored energy is less than 3J:

Stored potential energy $= \dfrac{AL\sigma\varepsilon}{2}$

$= ½\ (6000\ mm^2)\ (30mm)\ (0.12)(0.3)$

$< 3\ J$

So we don't need a more accurate value of Young's Modulus to see that running shoes make a negligible contribution to the stored elastic energy.

On a good running track, where the springiness is well tuned to the runners' stride patterns, such shoes might well be disadvantageous.

Summary

- Stress is defined as force/area
- Strain is defined as fractional change in length
- For simple elastic solids up to a strain of around 0.1, the relation between stress and strain is given by Hooke's law:

  stress $\propto$ strain.
- The coefficient of proportionality is Young's modulus

SAQs

1. During running the upward force on the femur is 6400 N. Its Young's modulus is 17 900 N mm^{-2}. It has a length of 500mm and a cross sectional area of 330 mm^2. How much energy to the nearest whole number of Joules is stored in the femur at maximum compression?

2. Using the data from question 1 how much shorter is the femur when you are you standing up?

   (a) no shorter because it is incompressible

   (b) 0.5 cm

   (c) 0.5 mm

3. Kangaroos have long tendons. What advantage does this give them?
   (a) They have a large stride length allowing them to cover more ground in less time
   (b) Their tendons will be under less stress so less likely to suffer injury
   (c) They can store more elastic energy allowing them to cover larger distances with less energy expenditure.

The answers appear on the following page

Answers

1. 2 Joules (The Potential energy is ½ x $(\text{strain})^2$ x volume / Y = 1.73 N m or about 2 J)

2. (a) Incorrect: All bodies are compressible to some extent (A truly incompressible body would have an infinite speed of sound.)

   (b) Incorrect: This is probably close to the total change in height, but not that of just the femur

   (c) Correct: From Hooke's law the change in length is $(6400\text{N}/330\text{mm}^2)\text{x}(500\text{mm})/17900\ \text{Nmm}^{-2} \sim 0.54$ mm

3. (a) Incorrect: animals with larger strides are not necessarily faster

   (b) Incorrect: Stress depends on body-weight and tendon thickness, not length. Take care not to confuse stress and strain. In fact, Kangaroo tendons have developed to resist high stresses

   (c) Correct: the elastic energy stored is proportional to length. Kangaroos recover over 50% of the energy during each hop

Structure of solids

What's wrong with a 60' gorilla?

More specifically, can we create a viable large creature by just scaling up a smaller one equally in all dimensions by a factor of, say, 10?

Image[2]

The answer isn't obvious, so here's a clue adapted from an article by J B Haldane:

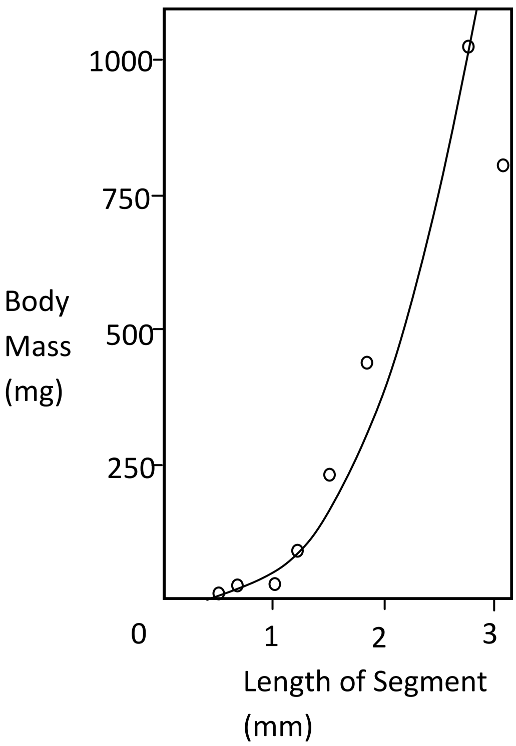
"The giants in Pilgrim's Progress, were ten times bigger than humans in every dimension, so their weight would have been a thousand times larger, say eighty tons or so. Their thighbones would only have a hundred times the cross section of a human thighbone, which is known to break if stressed by ten times the weight it normally carries. So these giants would break their thighbones on their first step." (J. B. S. Haldane)

What is the missing step in the argument?

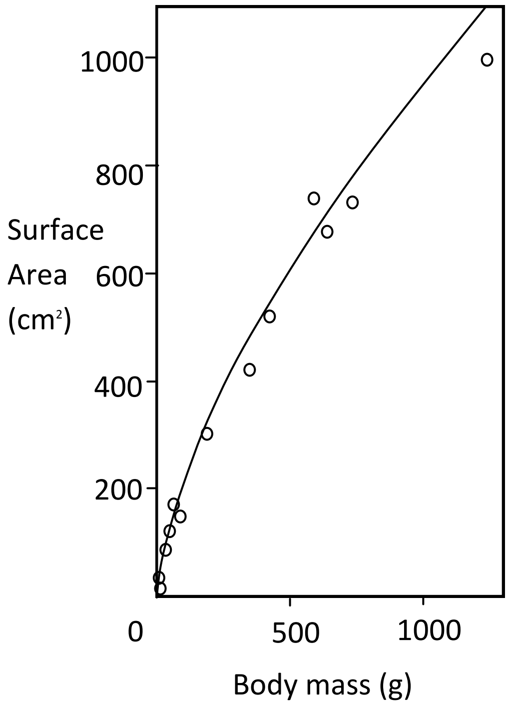
[2] KING KONG and the beauty, photograph by Keng Susumpow, as posted on www.flickr.com. Creative Commons Licensed.

Scaling

Let's look at a general argument. Suppose we scale up a body by a factor x in all dimensions: height, depth and breadth. The volume will increase by a factor of x^3, so the mass will increase also by x^3. This is shown for two examples in the graphs where the mass of cockroaches can be seen to scale with their length cubed and the surface area of salamanders with their mass to the 2/3 power.



Body mass, m, versus leg length, L, for the cockroach Periplaneta Americana. The curve is fit to individual specimens using $m=kL^3$.



Surface area versus body mass for dwarf sirens (salamanders). The curve is fit to specimens using $A=km^{2/3}$.

To break a bone we have to cleave it along a cross-section. If each molecule is bonded to its immediate neighbours we have to slice through a plane of bonds. The energy to do this is proportional to the number of bonds to be broken, hence increases as the cross sectional area of the bone, that is as x^2. However, the stresses put on the bone are proportional to the weight of the body or x^3. So as we increase x the stresses will rapidly outgrow the ability of the skeleton to support them. This is shown in the graph labelled figure A:

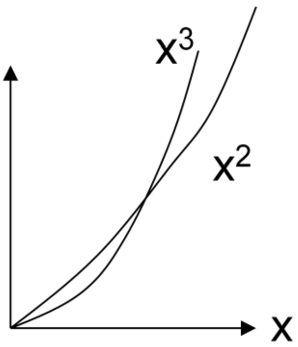
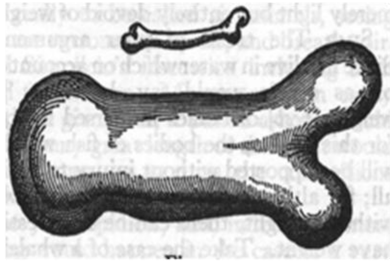


Figure A


To support an increase in weight bones must be relatively fatter

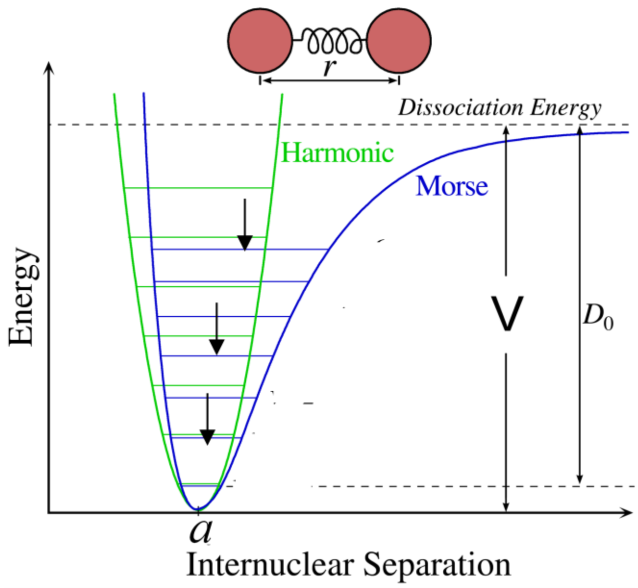
Looking at this the other way round, the area of a bone must increase in proportion to the mass of a body in order to support it. So the diameter of a bone must increase as $x^{3/2}$. Since the length is increasing as x the bone must get proportionally thicker. You can see this in the drawing of bones that Galileo made to illustrate the effect.

We can now see why the bones of the giants in Pilgrim's Progress would break as soon as they started to walk. To support a thousand times the weight we require bones a thousand times the area, but these are only 100 times the area. The stress is therefore 10 times bigger than in a human, which we are told is beyond what bones can support.

Strength of materials

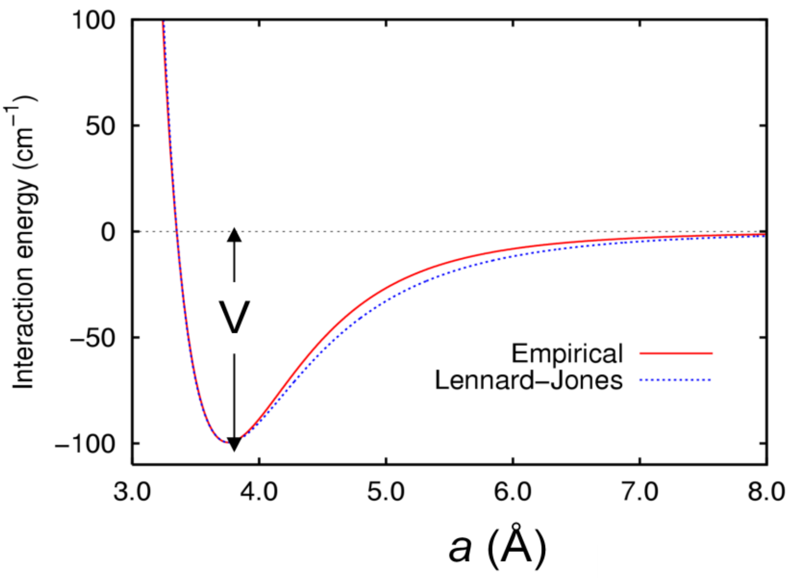
Let's look more closely at the forces between atoms to see how they determine the properties of materials. We assume that each atom or molecule in a crystalline solid attracts each of its immediate neighbours – and of course the attraction is mutual. The objective of this discussion is to see that the bulk properties of solids are related to this pair-wise interaction. There are various proposals for how the interaction can be modelled. They all involve a potential with a repulsive core, so that atoms can't get too close, and a diminishing attraction at large distances. We've plotted two examples of the potential energy as a function of separation: notice that they are both asymmetric.

Morse potential, 3 parameters *r*, *a*, *V*:



Image[3]

Lennard-Jones potential, 2 parameters *V*, the interaction energy, and *a,* the equilibrium atomic separation:



Image[4]

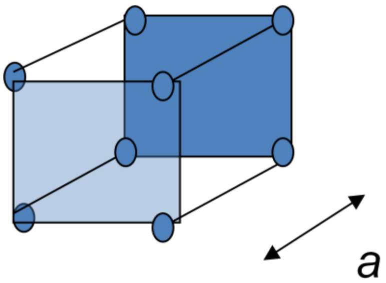
[3] Image downloaded from http://upload.wikimedia.org/wikipedia/commons/7/7a/Morse-potential.png. Creative Commons Licensed

This means that the atoms oscillate about a different mean position as they rise in energy: you can see that in the diagram of the Morse potential where different mid points are arrowed.

For our discussion we'll just use as our parameters the depth of the potential (V), or, equivalently, the interaction energy, and the distance between atoms (*a*) without worrying about the shape of the potential.

Bulk Properties

Here we show some estimates of various bulk properties in terms of the parameters of the interatomic potential *a* and *V.*



Let's start with Young's modulus. This is a force per unit area or dV/dr divided by an area. We can estimate dV/dr as V divided by the atomic separation a. The area occupied by an atom is a^2. So Young's modulus is of order of magnitude V/a^3. Another way to see this is to notice that force per unit area has the same dimensions as energy per unit volume: V gives us the potential energy associated with a pair of atoms in a volume of order a^3.

Turning next to the Latent Heat of sublimation we see that this involves an energy V to separate each pair of atoms. In a mole we have Avagadro's number N_A of atoms or N_A /2 pairs, so the latent heat is ½ N_A times V.

Finally the coefficient of thermal expansion is defined as δl/l times 1/T. Since the coefficient is approximately independent of T the T has to cancel out, so δl/l must involve a temperature. The Boltzmann's constant k converts a temperature to an energy, so δl/l will be dimensionless if we put it equal to kT divided by an energy. The only other energy involved

[4] Image downloaded from http://upload.wikimedia.org/wikipedia/commons/9/93/Argon_dimer_potential_and_Lennard-Jones.png. Creative Commons Licensed.

is the interatomic potential energy V. The coefficient of thermal expansion is therefore approximately Boltzmann's constant divided by V. This is the correct dependence, but only an approximate argument: for a harmonic potential the atomic oscillations are symmetric so the only affect of raising the temperature would be to increase the amplitude of oscillation not to expand the solid. Only if the potential is asymmetric does the equilibrium position move to a larger spacing as the energy increases, hence only for an asymmetric potential do we get thermal expansion. Look again at the arrows on the diagram of the Morse potential to see this.

Young's modulus ($N\ m^{-2} = Jm^{-3}$) = V/a^3 → 20 GPa

Latent heat of sublimation = energy mol^{-1} = $\frac{1}{2}VN_A$ → 154.7 kJ mol^{-1}

Coefficient of expansion = $\frac{\delta l}{l}\frac{1}{T} = \frac{kT}{\text{Energy}}\frac{1}{T} = \frac{k}{V}$ → 2.2×10^{-5} per degree

You can check these relations for calcium given that the interatomic spacing is 0.34 nanometres, and you can use data to determine the interatomic potential V for calcium.

Finally we can return to the strength of bones. Bones are mainly calcium, so to begin, let's estimate the amount of energy required to break a calcium crystal: This is the dissociation energy V times the number of atoms per unit area $1/a^2$ or Y times a.

Breaking stress ~ $V/a^2 = Ya$

This estimate comes out to be much lower than the stress required to break a bone which is around 1700 Joules per square metre. The difference arises because bones are not very well represented as calcium crystals – they have a complex molecular structure and are visco-elastic, that is they flow before breaking.

Summary

- Areas of organisms scale as $(\text{length})^2$ and masses as $(\text{length})^3$
- In order to resist the increased stresses the area of bones must scale as $(\text{length})^3$ not as $(\text{length})^2$ so must become relatively fatter for larger masses (other things being equal)
- We can predict the bulk properties of solids from the interatomic potential energy.

SAQs

1. Harry Potter's friend Hagrid is twice normal height and three times normal width. His bones are presumably twice the normal length. By what factor should their diameter be increased?

   (a) 18

   (b) $\sqrt{18}$

   (c) 3

2. What is the latent heat of sublimation of 1 kg of Calcium?

   (a) 4 kJ

   (b) 154.7 kJ

   (c) 8 MJ

   (d) 4 MJ

The answers appear on the following page

Answers

1. (a) Incorrect: this is the factor by which the volume changes

   (b) correct: the volume is increased by a factor 18 so this is the change in mass. Therefore this is the factor by which the area of the bones must increase to support this weight. The diameter or radius therefore must increase by $\sqrt{18}$

   (c) Incorrect: scaling the bones by the same linear factor will not provide sufficient strength to support the increased weight.

2. (a) Incorrect: you may have got the molecular weight wrong (a mole is 40g) or you've forgotten to convert from g to kg.

   (b) Incorrect: this is the heat of sublimation per mol not per kg

   (c) Incorrect: you may have used the atomic number (20) to define a mole instead of the correct atomic weight (40)

   (d) Correct: a mole of calcium is 40g (the molecular weight expressed in grams) so you just scale up the given values (154.7 kJ mol^{-1}) to 1 kg.

Surface Tension

The Problem

How do flies walk on water?

Image[5]

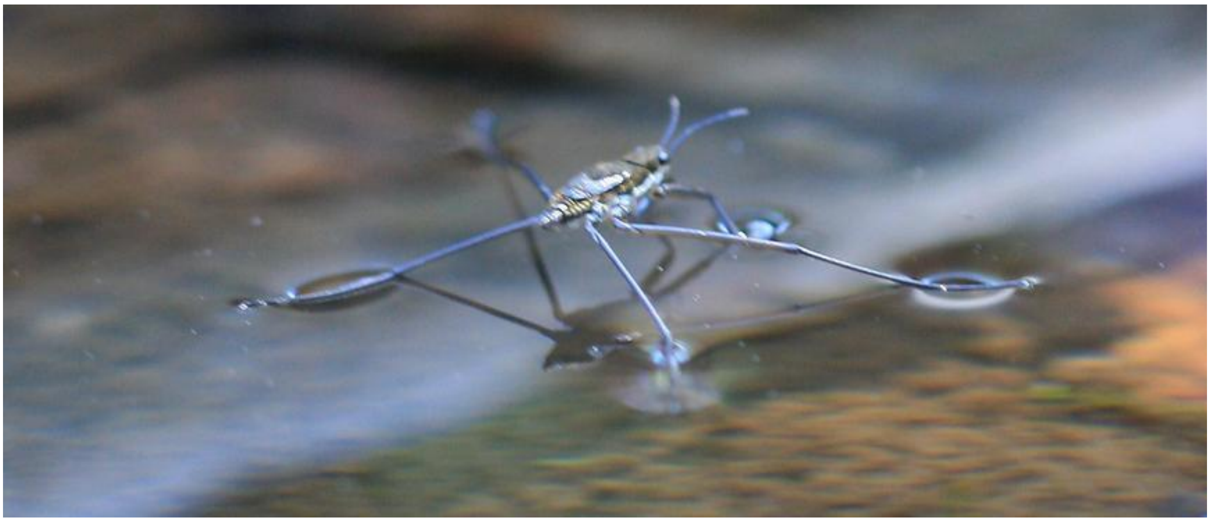
The water strider has characteristic length 1cm and weight 1-10 mg.

Try this with a paperclip. Look carefully and see if you can explain why it floats. (Hint: it has nothing to do with Archimedes)

Surface tension

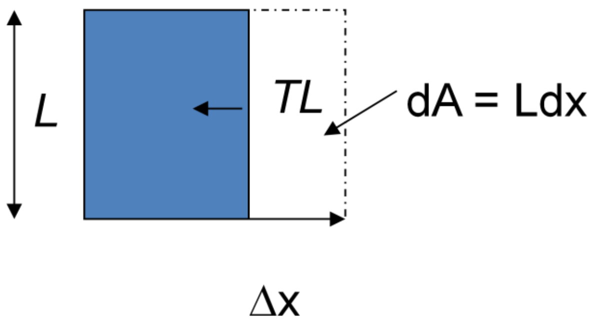
The close-up picture shows the depressions in the water where the water-strider makes contact with the surface. This suggests that the surface tension provides the support.

[5] DSC03688, by knickinoptik, as posted on www.flickr.com. Creative Commons Licenced.



Image[6]

Surface tension is defined as the force exerted parallel to the surface on a unit length of its boundary. So if the surface tension is T to increase the area of a length L of the boundary by an amount dx requires a quantity of work TLdx.



In the process the area increases by dA = Ldx. So the quantity of work done is TdA. Therefore, we can also define surface tension as the energy per unit area of the surface. From a molecular point of view, surface tension arises because the molecules at the surface are pulled on from one side only, namely the interior of the fluid, whereas the molecules in the interior bind to molecules on all sides. In order to increase the surface area molecules must come from the interior to the surface. To do this, molecular bonds must be broken which requires energy.

That is:

Surface tension = Force per unit length = energy per unit area

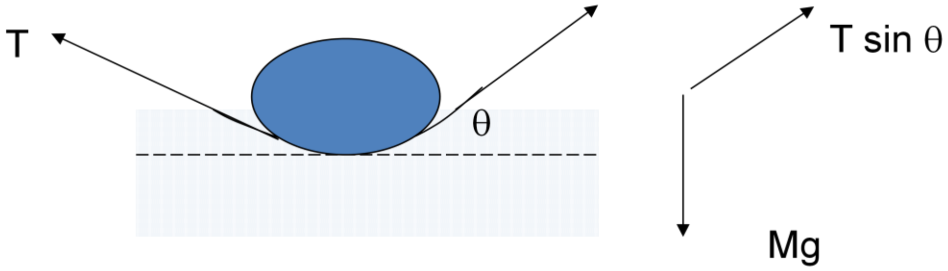
Work done $= F\Delta x = TL\,\Delta x = T\,\Delta A$

So surface tension is the work done in creating a unit area of surface.

[6] Water Strider, by Velo Steve, as posted on www.flickr.com. Creative Commons Licenced.

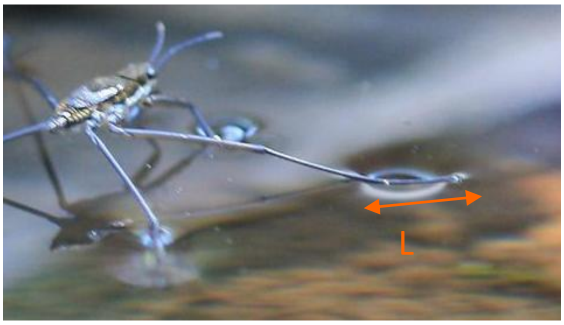
Walking on Water

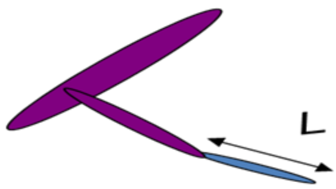
So insects can walk on water because the deflection of the surface produces a vertical component of surface tension which balances the weight of the insect. If the surface makes an angle θ with the horizontal, the upward force is TLsinθ over a length L on each side of the depression, so the condition for support of an object of mass M is Mg = 2TLsinθ.



$$Mg = 2TL\sin\theta \times (\text{number of feet})$$

Hairs on the legs of water-walkers also help prevent the water molecules from binding to the leg and hence flowing up around it.





Relation to observation

Many species of water strider have been extensively measured, and the relation between their weight and surface tension examined.. The force due to surface tension is always larger than the weight, since only the vertical component of surface tension supports the strider.

Smaller insects maintain a large safety margin, while the larger striders live close to the edge, where the two forces (vertical component of water tension and weight) become close to even.

Summary

Surface tension = work done per unit surface created

= force per unit length

SAQs

1. Water molecules are bound together with an average energy of order 0.1 eV or 1.6 x 10^{-10} J. There are around 6 x 10^{18} water molecules per unit area of surface? Estimate the surface tension of water? Give your answer in J m^{-2} to 1 s.f.
2. If the bodies of water striders are scaled up in size by a factor L, the body weight will scale by a factor L^3. How will the force due to surface tension scale?

   (a) L^3

   (b) L^2

   (c) L

3. Why does water from spherical droplets in zero gravity? (Choose as many as apply)

   (a) Because this gives the minimum surface area hence the minimum energy

   (b) Because there is no up-down direction in zero g so the form of the water droplet must be symmetrical

   (c) Because the force due to surface tension must balance across any line in the surface so the angle to the tangent plane must be the same everywhere

4. Why does water form into droplets on a greasy surface?

   (a) Because water cannot flow across a greasy surface

   (b) Because the grease attracts the water

   (c) Because the grease repels the water

   (d) Because water forms droplets which are unaffected by the surface

The answers appear on the following page

Answers

1. 0.1 Correct: The surface tension is of order the energy per bond times the number of molecules per unit area, hence 1.6 x 10^{-10} J x 6 x 10^{18} molecules m^{-2}. The correct value is 0.072 J m^{-2}.

   Other answers: Incorrect: The surface tension is of order the energy per bond times the number of molecules per unit area, hence 1.6 x 10^{-10} J x 6 x 10^{18} molecules m^{-2}. The correct value is 0.072 J m^{-2}.

2. (a) Incorrect: If this were true then insects of any size with the form of water-striders, however large, could walk on water.

   (b) Incorrect: you may be thinking that the surface tension is proportional to area, but we are asking for the force this generates on the legs of the insects.

   (c) Correct: the surface tension is the same for all insects, but the force depends on the length of the leg. The surface tension gives a force $\propto$L. This suggest a relation between surface tension and weight of the form F_s µ $F_g^{1/3}$. This is represented as the dashed line in the graph comparing theory and observation for species of water-striders. The fact that the best fit line has a slope larger than 1/3 shows that the legs of large water striders are proportionally longer.

3. (a), (b) and (c) are all correct explanations

4. (a) Incorrect. There is some truth in this – clearly a small droplet does not flow – but a large enough one would so it's not the whole explanation.

   (b) Incorrect. Even if grease did attract water, which it doesn't, this would tend to spread the water out across the surface, not confine it to droplets.

   (c) Correct: This is the inverse behaviour to the formation of a meniscus in a tube. The repulsion tends to confine the droplet.

   (d) Incorrect: The water in the droplet will flow unless the surface properties prevent it.

Fluid Flow

The Problems

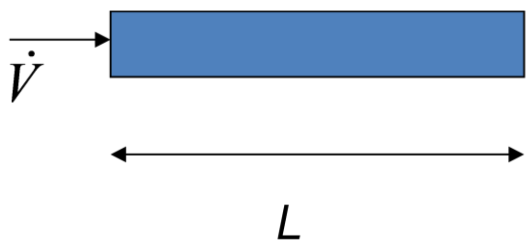
1. Blood vessels dilate when exercising to carry away excess heat. The rate of blood flow varies by a factor of around 3. Why is only a small dilation of by a factor of order 1.3 sufficient?

2. How does water get up a tall tree?

Flow in a pipe

To address the flow of blood we think of a viscous fluid in a pipe. How do you think that increasing the pressure at one end of the pipe will alter the flow rate? How will a change in viscosity affect the flow rate? Think about this for a moment before going on.

A fluid flows faster the harder it is pushed and more slowly the more viscous it is. So the flow rate will be proportional to the pressure gradient along the pipe and inversely proportional to the viscosity. Knowing this we can use dimensional analysis to find the dependence of the flow rate on the other factor, namely the radius of the pipe. We saw how to do this in session 1.



Let the flow rate, V dot, be proportional to the pressure gradient $\Delta P/L$, inversely proportional to the viscosity η and proportional to some unknown power α of the radius *a* of the pipe.

$$\dot{V} \propto \frac{a^{\alpha}}{\eta} \frac{\Delta P}{L}$$

Units: η viscosity = Nsm^{-1} = MT^{-1}

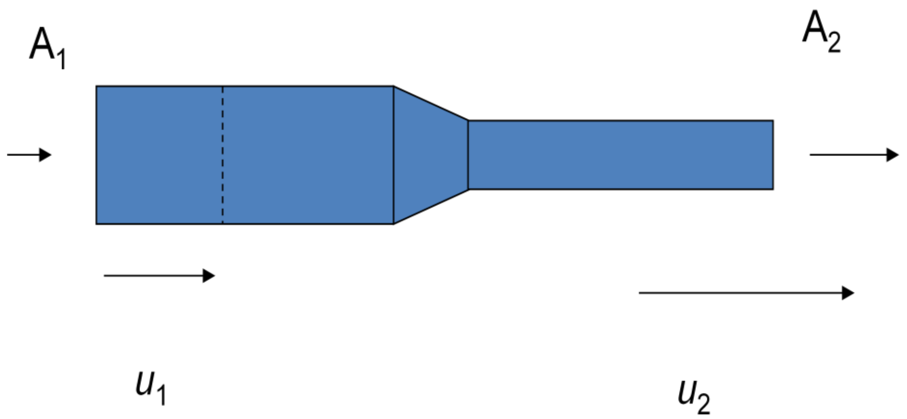
P pressure = $Nm^{-2} = MT^{-2}$

The units of the flow rate are m^3 per second, hence Length^3 timesTime^{-1}. $\Delta P/\eta$ is Time^{-1}. So we just have to balance the units of length. This gives $\alpha = 4$. Stop for a moment to check this before moving on.

What does the result mean for our problem of the flow rate of blood in arteries? To change the flow rate by a factor 2 we need only expand an artery by a factor of $2^{1/4}$ or about 1.2, an increase of 20% not 100%.

Conservation of mass

Although it's not central to the problem this is a good point to think about what happens to the rate of flow if a pipe narrows. The mass passing a given point must remain constant in a steady flow, since there can be no piling up of mass.



Look at the left hand end of the pipe, where the area is A_1, the speed of the fluid u_1 and its density is ρ_1. In one second a length u_1 of fluid enters the pipe, and this has a mass $\rho_1 A_1 u_1$. This quantity must be the same everywhere along the pipe, which gives us the equation of conservation of mass, equation (1). A fluid is incompressible if its density is constant so $\rho_1 = \rho_2$. In this case equation (1) simplifies to equation (2).

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 \qquad (1)$$

Incompressible:
$$A_1 u_1 = A_2 u_2 \qquad (2)$$

We can see from this that at a constriction the rate of flow *speeds up*.

How does water get up a tree?

Our next problem concerns the rising sap in plants. We first have to see what the problem is.

Suppose we divide two solutions of different concentrations of solute by a permeable membrane across which solvent can flow. We expect there to be a net flow of solvent in a direction that reduces the difference in concentrations, so from the more concentrated side to the less concentrated. This flow can be associated with a pressure that drives it which we call Osmotic Pressure.

The osmotic pressure between pure solvent and a dilute solution is given by a formula that resembles the gas law: the pressure is the concentration times the gas constant times the absolute temperature.

Osmotic pressure: $P = cRT$

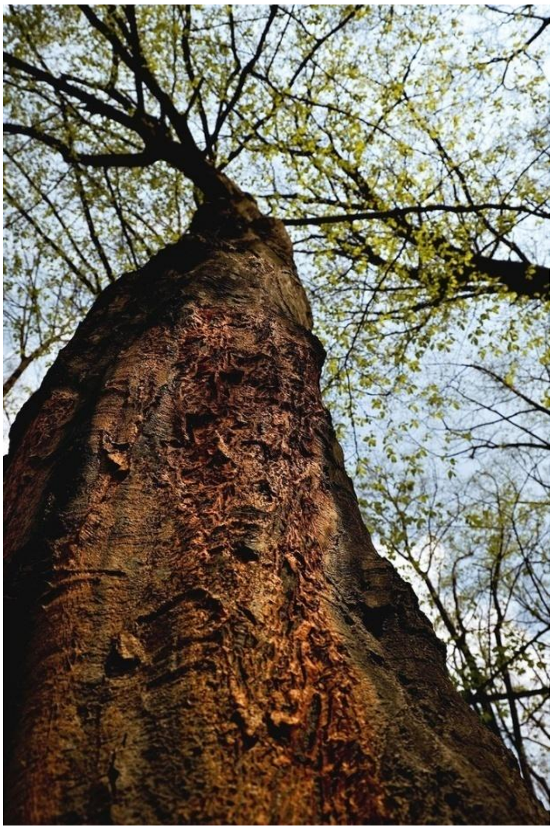
$R = 8.3$ kPa litres mol^{-1} K^{-1}

The units of concentration have to match those of the gas constant for this formula to work: normally the concentration is expressed in moles per litre and the gas constant in (pressure units – litres- per mole- per degree). For example if the units of pressure are kilo Pascals the gas constant is 8.3 kilo Pascals litres per mole per degree. These are different units from those we used when dealing with gases.

For a 0.1 molar concentration the pressure is about 240 kilo Pascals or 2.4×10^5 N m^{-2}. For a 100m tall tree the required pressure is 10^6 Nm^{-2} or 1000 kilo Pascals.

For a 0.1 molar solution
$P = 0.1 \times 8.3 \times 300 = 240$ kPa
For a 100 m tree:
$P = \rho g h = 1000 \times 10 \times 100 = 10^3$ kPa
So this is not the explanation

So now we have our problem: osmotic pressure is not sufficient to sustain a column of sap in a tall tree.



Image[7]

Pulling not pushing

Let's look at the problem the other way round: suppose instead of pushing from below on a column of water we are pulling from above. How tall could the column be? If you're confused by the question, think of a column of solid freely suspended under its own weight. The column will break when the weight exceeds the tensile strength. Water in a pipe also has a tensile strength which is surprisingly high, about 5 Mega Pascals or 50 atmospheres. This will support a column 500 m tall, sufficient for the tallest tree.

Tensile strength of water = 50 at. = 5×10^6 N m^{-2}

This will support a column for which $\rho gh = 5 \times 106$ N m^{-2}

or h = 500 m

The pull on the column comes from the evaporation through the leaves of the tree. But we haven't quite solved the problem: The water gains potential energy in rising up the tree.

[7] Strange burnt patterns on tree trunk, by Horia Varlan, as posted on www.flickr.com. Creative Commons Licensed.

Where does this energy come from? It's not a particularly easy question: you need to think back to thermodynamics and what we learnt about entropy.

Note: A fully grown tree may lose several hundred gallons (a few cubic meters) of water through its leaves on a hot, dry day.

Summary

- The flow rate of a viscous liquid in a pipe is given by Poiseulle's formula

$$\dot{V} \propto \frac{a^{\alpha}}{\eta} \frac{\Delta P}{L}$$

- Conservation of mass implies $\rho A v$ = const
- Incompressible flow: ρ = constant
- Osmotic pressure is given by $P = cRT$

SAQs

1. Which of the following will (in theory) accomplish a change by a factor of 4 in the rate of flow of a viscous fluid through a pipe? (Tick as many as apply)

   (a) Increase the length by 4

   (b) Increase the pressure at one end by a factor 2 and decrease the pressure at the other end by a factor of 2

   (c) Decrease the viscosity by a factor of 4

   (d) Increase the diameter by a factor of 2

2. A saturated salt (NaCl) solution at 273 K contains 6.1 mols of salt for every litre of solution. Its osmotic pressure is

   (a) 14 kPa

   (b) 14 MPa

   (c) 15 kPa

3. Assuming blood to be incompressible, by what factor is the speed of blood flow changed in a constricted artery the diameter of which has been reduced by ½ ?

   (a) ¼

   (b) ½

   (c) 2

   (d) 4

The answers appear on the following page

Answers

1. (a) Incorrect: increasing the length decreases the pressure gradient and hence the flow rate. It only serves to increase the stock of fluid in the pipe by a factor of 4.

   (b) Incorrect: this combination is not equivalent to changing the pressure gradient by a factor of 4. For example, if the pressure were 1.5 and 1 atmosphere initially, giving a difference of 0.5, they would become 3 and ½ giving a difference of 2.5 which is not 4 times 0.5.

   (c) Correct: form Poiseuille's formula the rate of flow is inversely proportional to viscosity

   (d) Incorrect: the diameter (or radius, a) must change by a factor of $2^{1/2}$ to give a change in a^4 by a factor $(2^{1/2})^4 = 4$.

2. (a) Correct. We use P = cRT = 6.1 x 8.3 x 273 = 13822 Pa

   (b) Incorrect. You nay have used m^3 instead of litres (or dm^3)

   (c) Incorrect. The temperature here is 273K not the 300 K in the example we gave.

3. (a) Incorrect: the speed increases at a constriction to keep vA = constant

   (b) Incorrect: the speed increases at a constriction to keep vA = constant

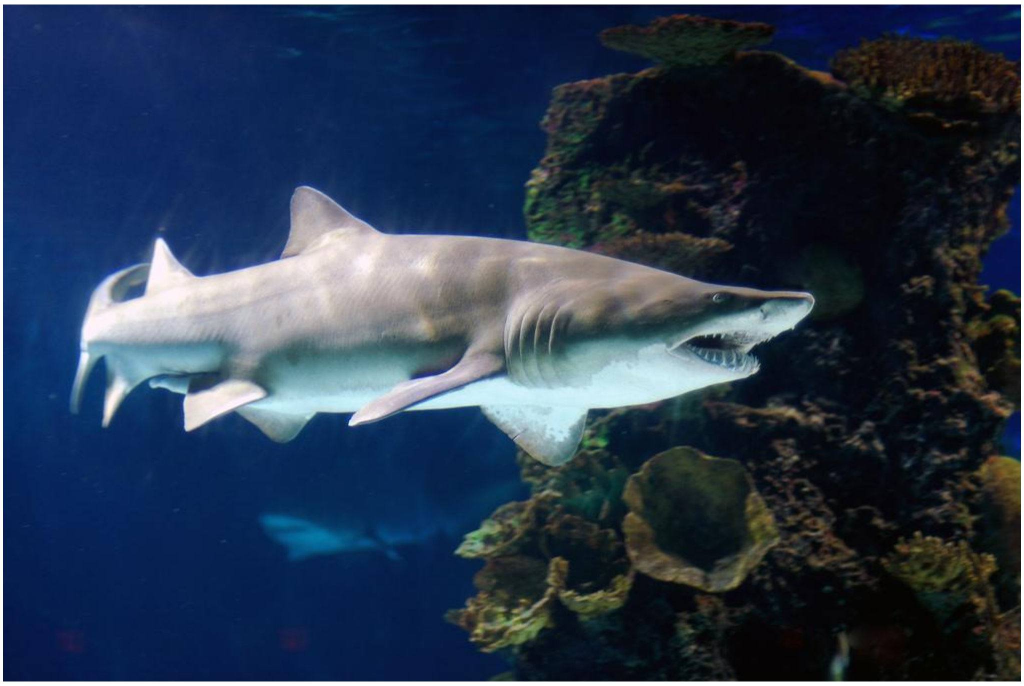
   (c) Incorrect: the diameter changes by 1/2 so the area is reduced to ¼ and hence the speed increases by a factor 4, not 2, to keep vA = constant

   (d) Correct: the diameter changes by 1/2 so the area is reduced to ¼ and hence the speed increases by a factor 4 to keep vA = constant

Buoyancy & Bernoulli

The Problem

Why do sharks never sleep?



Image[8]

A typical adult great white shark measures 4 to 4.8 m with a typical weight of 680 to 1,100 kg.

Archimedes' Principle

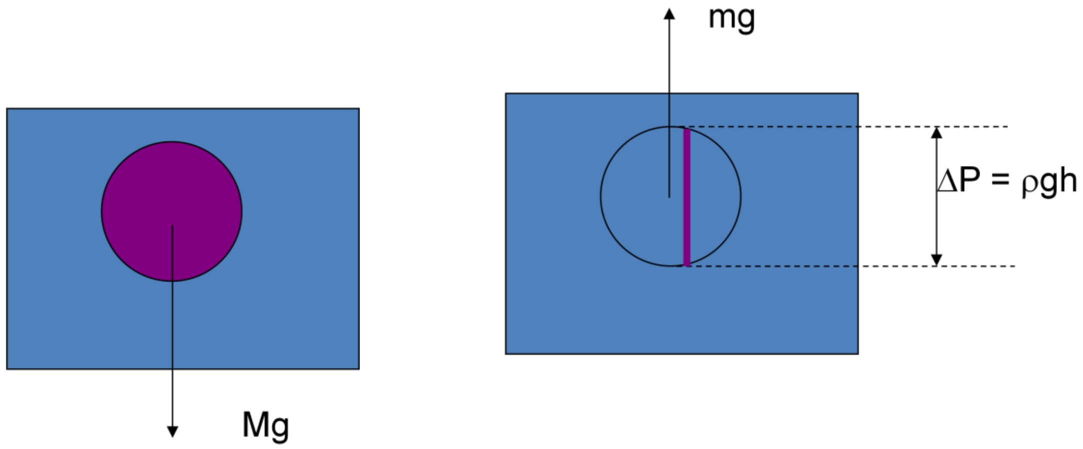
If sharks never sleep, the question that might occur to you is, what do they do during all their waking hours? It turns out that they swim most of the time. We need to find out a little more of the physiology of sharks in order to understand why this is.

It turns out that sharks do not have a swim bladder. This is the organ which the majority of fish use to maintain buoyancy or to change their depth in the water. How do they do this? The amount of gas in the swim bladder and the external pressure together determine the

[8] Shark, by Jeff Kubina, as posted on www.flickr.com. Creative Commons Licensed.

volume of the fish. As the volume changes the amount of water displaced changes hence, according to Archimedes' principle, so does the upthrust. Thus, the fish can use its swim bladder to control its vertical movement.

Archimedes' principle is illustrated below. The pressure difference between the top and bottom of the submersed object equals the weight of the column of ambient *fluid* and this is what provides the upthrust on each column of the body. The total upthrust is therefore the weight of the fluid that the body displaces.



Bernoulli's Theorem:

We know that a shark cannot maintain its buoyancy in this way, by changing its volume, so how does the fact that it is constantly swimming achieve this? The forward motion must provide some lift; this reminds us of flight, so how does that work? From the viewpoint of the shark, the sea is flowing over its fins. If this creates lift it must be because there is a pressure difference between the top and bottom surfaces of the fins. How does a pressure difference arise in a moving fluid?

For this we need Bernoulli's theorem. If a fluid is moving smoothly we can trace the particle motion as smooth curves called streamlines. If we follow a parcel of incompressible fluid (that is, a small volume) along a streamline, we find that the change in its energy is equal to the work done on it, or equivalently, the sum of the change in energy and the work done by the fluid must be zero. This gives us Bernoulli's theorem.

Incompressible, so ρ = *constant;* ΔV = *constant;* Av = *constant.*

Since the fluid is incompressible its volume is unchanged and so too is its density. Consider therefore the motion of a small volume ΔV. The kinetic energy of this volume on entry is $\frac{1}{2} m u_1^2$ or $\frac{1}{2}\rho \Delta V$ times u_1^2 . So in the table below, the change in kinetic energy is $\frac{1}{2}\Delta V$ times $(\rho_2 u_2^2 - \rho_1 u_1^2)$. The potential energy changes by $\rho \Delta V$ times $g\Delta h$ where Δh is the difference in heights at the outlet and inlet, $h_2 - h_1$, say.

The work done in pushing a volume ΔV through the pipe at the entry point just the pressure force times distance or $P_1 A_1 dx_1$ or $P_1\Delta V$. At the exit point the pressure opposes the motion so the work done on the gas is $-PA_2dx_2$ or $-P_2\Delta V$. The work done by the gas just has the signs reversed.

Change in KE	$\frac{1}{2}\rho\Delta V u_2^{\,2} - \frac{1}{2}\rho\Delta V u_1^{\,2}$
Change in PE	$\rho\Delta V g\Delta h = \rho\Delta V g(h_2 - h_1)$
Work done by the fluid	$\Delta V(P_2 - P_1)$

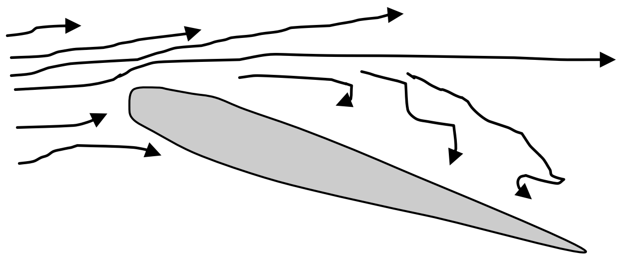
Setting the change in energy plus the work done by the gas to zero we get Bernoulli's theorem.

Bernoulli's Theorem: $\frac{1}{2}\rho v^2 + P + \rho g h = \text{constant}$ **along a streamline**

Notice that we can draw an immediate conclusion that at a given height a higher speed will correspond to a lower pressure.

Flow over a Wing

The real explanation is NOT the extra path theory! There is a lot of confusion around how to apply Bernoulli's theorem to the generation of lift by an aerofoil. The crude idea sometimes put forward is that the length of travel over the top surface of a wing is longer than over the bottom surface and the consequently higher speed of the airflow over the top surface generates a lower pressure and hence lift. This is quite wrong, since aircraft can fly upside down. However, Bernoulli's theorem still applies and the average airflow *is* faster over the topside of the wing, although not for this reason. In fact, the flow is quite complicated as can be seen from the picture below, which shows a tracing of smoke patterns illustrating how the air moves over a wing. Overall, the wing imparts a downward component of momentum to the air flow which corresponds to an upward force on the wing.



The complicated flow patterns makes it difficult to calculate the lift but Bernoulli's theorem can be applied to estimate the maximum lift as we'll see in the next section.

The Shark

Now back to the shark. The shark swings its tail to swim which forces water over its fins. The flow over the fins creates lift as in an aircraft. The shark can also tilt its fins to create a downward force instead.

Are we satisfied with this solution? Perhaps we should check it quantitatively. A shark might typically swim at 8 km hr^{-1} or very roughly about 2 m s^{-1}. Let's say the surface area of fins for the great white shark is 1 m^2. Using Bernoulli's theorem, the maximum possible pressure difference across its fins would be ½ times the density of water times its speed squared or 2000 N m^{-2}.

$$= \frac{1}{2}\rho v^2 \quad = 2000 \text{ N m}^{-2}$$

So one square metre of fin would support a 200kg mass. The shark is more massive than this and in practice the lift would be well short of this theoretical maximum. However, most of its weight is supported by buoyancy so the lift generated by the fins needs to support only a small fraction. This therefore seems entirely reasonable.

Summary

Bernoulli's theorem for streamline flow states that along a streamline

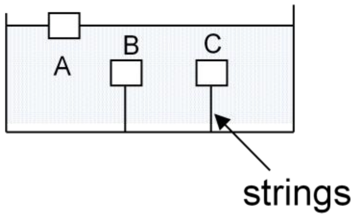
$$\frac{1}{2}\rho v^2 + P + \rho g h = \text{constant}$$

SAQs

1. Blocks A, B and C in the figure have same size and shape and are immersed in a fluid; the mass of A = mass of B < mass of C.

   (i) Is the buoyancy force on A (a) > , (b) = , or (c) < the force on B

   (ii) Is the buoyancy force on B (a) >, (b) = , or (c) < the force on C?



2. A bucket has a hole 20 cm below the water surface.

   (i) How fast is water flowing through the hole? [Express your answer in an integer number of m s^{-1}.]

   (ii) If the hole is $10mm^2$ in diameter, what is the rate of egress of the water? [Express your answer in an integer number of grams / sec]

   (iii) If the water continues to flow at this rate the bucket will empty in 5 minutes. Is the actual time taken (a) longer (b) shorter (c) exactly this

3. Assuming blood flow to be incompressible with density ρ, speed v, by how much does the blood pressure change in a constricted artery the area of which has been reduced to 1/3 ?

   (a) $- 4\rho v^2$

   (b) $-2/3\ \rho v^2$

   (c) $+ 2/3\ \rho v^2$

   (d) $+ 4\rho v^2$

The answers appear on the following page

Answers

1. (i)
   (a) Incorrect: the fact that it is floating does not mean the upthrust (or buoyancy force) is bigger. The force just depends on the weight of fluid displaced.
   (b) Incorrect: Perhaps you are thinking that B is the same mass as A so the forces must be equal. But the buoyancy force just depends on the weight of fluid displaced which is not equal.
   (c) Correct: A displaces less fluid so experiences a smaller upthrust.
   (ii)
   (a) Incorrect: you may be reasoning that the lighter object is more likely to float so should be experiencing a greater push to the surface. But the upthrust can only come from the displaced fluid which is equal in the two cases.
   (b) Correct: the weights of displaced fluid are equal so the buoyancy forces are equal
   (c) The less massive body does not experience a smaller upthrust because this depends only on the weight of fluid displaced, which is the same in both cases.

2. (i) 2. Use Bernoulli's theorem to show that the speed is $v = (2gh)^{1/2}$ for a depth h = 20 cm
   (ii) 20. Use rate of flow = density x speed x area
   (iii)
   (a) Correct: the speed of egress decreases as the head of pressure drops
   (b) Incorrect: it might look as if a conical bucket empties more quickly at the end but the speed of egress decreases as the head of pressure drops so it will actually take longer than 5 minutes.
   (c) Incorrect: the only difference between the estimated 5 minutes and the actual time is the drop in speed of egress of the water as the head of pressure falls according to Bernoulli's theorem.

3. (a) Correct: apply Bernoulli's theorem to the flow upstream of the constriction and at the constriction.
   (b) Incorrect: in applying Bernoulli's theorem you may have forgotten that it involves v^2 µ A^2.
   (c) Incorrect: you may mistakenly used A instead of A^2 in substituting for v^2 in Bernoulli's theorem, but you have also not kept track of the signs: the drop in area creates a more rapid flow which leads to a pressure drop
   (d) Incorrect: You have also not kept track of the signs in Bernoulli's theorem: the drop in area creates a more rapid flow which leads to a pressure drop

Meta tags

Author: Sarah Symons.

Owner: University of Leicester

Title: Enhancing Physics Knowledge for Teaching – Properties of matter

Keywords: Elasticity; Structure; Surface tension; Fluid flow sfsoer; ukoer

Description: In this session we'll look at some of the properties of matter.

Language: English

Version: 1.0

Additional Information

This pack is the Version 1.0 release of the module. Additional information can be obtained by contacting the Centre for Interdisciplinary Science at the University of Leicester. http://www.le.ac.uk/iscience